Project-Physics of raindrops

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As a raindrop falls through a stationary mist of suspended water droplets, its speed increases and it grows in size. Assume that the rain drop retains a spherical shape as it falls and grows. It is found empirically that the drops attain a 'settling speed' which is related to the drop radius (the final speed and radius of the drops when they emerge from the mist). That is, if you find a raindrop with a certain radius, it is likely to have a certain velocity that depends on this radius (seemingly independent of its history of formation and path through the mist). This project aims to investigate this phenomenon.

We model the problem as follows. As the drop falls, it accumulates mass. In addition, it experiences a drag force due to the air. Determine an expression for the rate of change of mass of the rain drop as a function of its instantaneous radius and velocity. The drag force is a little complicated, with its magnitude given by

$$F_D = \frac{C}{2}\pi\rho_a r^2 v^2$$

where v is the speed of the drop and r its radius. ρ_a is the density of air. This expression is not difficult to justify, so long as C is a constant. Assume a model in which the air molecules are stationary and just bounce off from the spherical raindrop as it moves through the air. Argue that just given this, the drag force should be of the form as above.

However, since the air molecules are in motion, it turns out C is not constant, but depends on the drop radius, its speed and the viscosity of air ν through a dimensionless number called *Reynolds number*, given by

$$R = \frac{2rv}{\nu}$$

In the range $10 < R < 1000$, this dependence is seen to agree with the relationship

$$C = 12R^{-1/2}$$

Write simultaneous equations for the rate of change of speed of the drop and the rate of change of its radius. You should get equations of the form

$$\begin{aligned}\frac{dv}{dt} &= g\left[1 - \left(\frac{v}{\sigma r}\right)^{3/2}\right] - 3\epsilon\frac{v^2}{r}\\ \frac{dr}{dt} &= \epsilon v\end{aligned}$$

where g is acceleration due to gravity and the parameters σ and ϵ depend on the various densities, viscosity of air and acceleration due to gravity.

Determine the numerical values (and dimensions) of σ and ϵ, given the following data for density of air ρ_a, density of the mist ρ_m and density of water ρ_w and the viscosity of air ν relevant for a typical cloud

$$\begin{aligned}\rho_a &= 0.856\times10^{-3}\text{ g cm}^{-3}\\ \rho_w &= 1\text{ g cm}^{-3}\\ \rho_m &= 1\times10^{-6}\text{ g cm}^{-3}\\ \nu &= 0.206\text{ cm}^2\text{s}^{-1}\end{aligned}$$

Argue that far as the change in speed of the drop is concerned, the dominant effect is that of air drag. The 'mist drag' (effective damping force due to increase in mass of the drop) is a small effect, though it is important to account for the increase in size of the drop. First, ignore the mist drag term and analyze the change in velocity if a water drop of radius r_0 that starts from rest (the radius does not change). Without explicitly solving the equation, argue that the drop will reach a terminal velocity. Determine this velocity. Next, numerically determine the speed of the drop as a function of time and plot the variation. Identify suitable natural length and time scales (which also determine a velocity scale) and express the relevant quantities in units of these scales to solve the equation. For simplicity, try using the following simple algorithm (Euler's Algorithm)

$$\begin{aligned} \text{Given } \frac{dv}{dt} &= f(v) \\ v(t+\Delta t) &\approx v(t) + f(v)\Delta t \end{aligned}$$

Next, solve the full problem. Starting with different initial drop sizes (different values of r_0 varying from 0.1 mm to 0.5 mm) and for drops starting at rest, determine the variation of speed and radius of the drop as a function of time, upto the time at which the final radius of the drop is 1 mm (after this size, the assumptions made in this model start breaking down. The drop willl start flattening and will not stay spherical). Plot these variations. Figure out the simplest algorithm to solve these equations (given radius and velocity at an instant, determine these infinitesimally later).

Finally, for each initial drop size, plot the variation of speed of the drop with the radius of the drop. You should observe something interesting about the relationship between the settling speed and the radius of the drop.