Applications of Fourier Integral Transform

A. Gupta

[1] Department of Physics
St. Stephen's College

Outline

Fourier Integral Transform

Fourier Integral Transform:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \tilde{f}(k)\, e^{ikx}$$

Inverse Transform:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, f(x)\, e^{-ikx}$$

Fourier Integral Representation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} dx'\, f(x')\, e^{-ikx'}$$

Fourier Integral Representation of Dirac Delta Function

Fourier Integral transfrom of the Dirac Delta function:
$f(x) = \delta\,(x - x_0)$

$$\begin{aligned} \tilde{f}(k) &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, \delta\,(x - x_0)\; e^{-ikx} \\ &= \frac{1}{\sqrt{2\pi}}\; e^{-ikx_0} \end{aligned}$$

It then follows that

$$\begin{aligned} \delta\,(x - x_0) &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \tilde{f}(k)\; e^{ikx} \\ &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \frac{1}{\sqrt{2\pi}}\; e^{-ikx_0}\; e^{ikx} \end{aligned}$$

Fourier Representation of Dirac Delta Function

$$\delta\left(x - x_0\right) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ e^{ik(x - x_0)}$$

Problem

Consider the sequence of functions

$$\delta_n(x) = \begin{cases} n, & |x| < 1/2n \\ 0, & |x| > 1/2n \end{cases}$$

Expressing $\delta_n(x)$ as a Fourier Integral, show that

$$\begin{aligned} \delta(x) &= \lim_{n\to\infty} \delta_n(x) \\ &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ e^{ikx} \end{aligned}$$

(A few) Postulates of Quantum Mechanics

1. The state of a particle is decribed by a square-integrble complex function, the *wavefunction* $\psi(x)$.
2. If a measurment of position is made, the probability of detecting the particle between x and $x + dx$ is
$$P(x)\ dx = \frac{|\psi(x)|^2}{\int_{-\infty}^{\infty} dx\ |\psi(x)|^2}\ dx$$
3. The result of momentum measurement is encoded in the 'momentum-space wavefunction' $\tilde{\psi}(p)$ which is the Fourier Integral Transform of $\psi(x)$
$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} \psi(x)\ e^{-ipx/\hbar}$$
where $\hbar = h/2\pi$. If the momentum of the particle (whose wavefunction is $\psi(x)$ is measured, the probability of measuring momentum between p and $p + dp$ is given by
$$\tilde{P}(p)\ dp = \frac{\left|\tilde{\psi}(p)\right|^2}{\int_{-\infty}^{\infty} dp\ \left|\tilde{\psi}(p)\right|^2}\ dp$$
4. The wavefunction eveolves with time according to Schrodinger Equation
$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

Position vs Momentum

Wavefunction for a state of well-defined position x_0:

$$\psi_{x_0}(x) = \delta(x - x_0)$$

Note this is not square-integrable! What if we measure momentum of the particle?

$$\begin{aligned}\tilde{\psi}_{x_0}(p) &= \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}} \, \delta(x - x_0) \; e^{-ipx/\hbar} \\ &= \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx_0/\hbar}\end{aligned}$$

For this $\left|\tilde{\psi}_{x_0}(p)\right|^2 = 1/(2\pi\hbar)$. Therefore probability of any momentum is the same! However, strictly, $\tilde{P}(p)$ is not defined (why?)

Wavefunction for a state of well-defined momentum p_0:
The momentum-space wavefunction should be

$$\tilde{\psi}_{p_0}(p) = \delta\left(p - p_0\right)$$

The fourier transform gives the position space wavefunction as

$$\psi_{p_0}(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ip_0 x/\hbar}$$

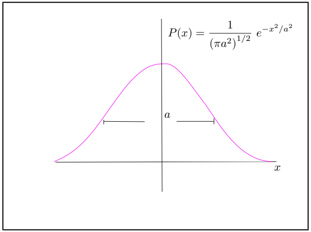
For this, $|\psi(x)|^2 = 1/\left(2\pi\hbar\right)$ is a constant. Though strictly, $P(x)$ is not well-defined.

Gaussian Wavefunction

$$\psi(x) = \frac{1}{(\pi a^2)^{1/4}}\, e^{-x^2/2a^2}\, e^{ip_0 x/\hbar}$$

Position probability distribution:

$$P(x) = \frac{1}{(\pi a^2)^{1/2}}\, e^{-x^2/a^2}$$



Is the complex phase $e^{ip_0 x/\hbar}$ superflous? No, it contains momentum information!

Momentum wavefunction:

$$\begin{aligned}\tilde{\psi}(p) &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}} \psi(x)\, e^{-ipx/\hbar} \\ &= \frac{1}{(\pi a^2)^{1/4}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}}\, e^{-x^2/2a^2}\, e^{ip_0 x/\hbar}\, e^{-ipx/\hbar}\end{aligned}$$

Clearly,

$$\tilde{\psi}(p + p_0) = \frac{1}{(\pi a^2)^{1/4}} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}}\, e^{-x^2/2a^2}\, e^{-ipx/\hbar}$$

which involves Fourier transform of a Gaussian function.

Fourier Transform of Gaussian

We need to evaluate the integral

$$f(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-\alpha x^2}\, e^{-ikx}$$

where $\alpha = 1/\left(2a^2\right)$ and $k = p/\hbar$. Complete the square in the exponent

$$-\alpha x^2 - ikx = -\alpha \left(x + \frac{ik}{2\alpha}\right)^2 - \frac{k^2}{4\alpha}$$

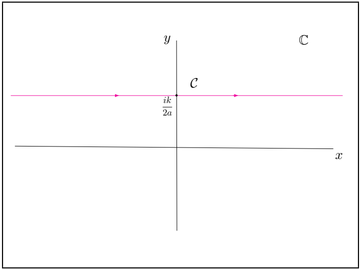
The integral becomes

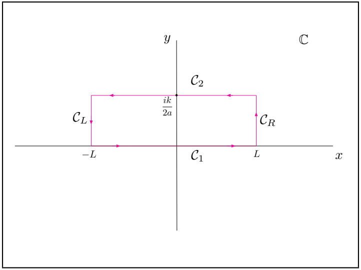
$$f(k) = \frac{1}{\sqrt{2\pi}}\, e^{-k^2/4\alpha} \int_{-\infty}^{\infty} dx\, e^{-\alpha\left(x + \frac{ik}{2\alpha}\right)^2}$$

$$I = \int_{-\infty}^{\infty} dx \; e^{-\alpha\left(x+\frac{ik}{2\alpha}\right)^2}$$

Visualisation in the complex plane:

$$I = \int_{\mathcal{C}} dz \; f(z), \quad f(z) = e^{-\alpha z^2}$$





$$\oint_{\mathcal{C}} dz\, f(z) = \int_{\mathcal{C}_1} dz\, f(z) + \int_{\mathcal{C}_2} dz\, f(z) + \int_{\mathcal{C}_L} dz\, f(z) + \int_{\mathcal{C}_R} dz\, f(z)$$
$$= 0$$

As $L \to \infty$, $\int_{\mathcal{C}_R} dz\, f(z) = \int_{\mathcal{C}_L} dz\, f(z) \to 0$. Therefore

$$I = \lim_{L\to\infty} \int_{\mathcal{C}_1} dz\, f(z)$$
$$= \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}$$
$$= \sqrt{\frac{\pi}{\alpha}}$$

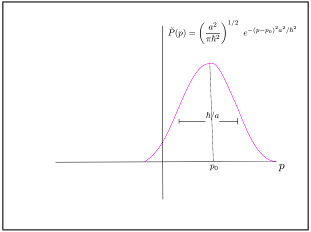
Then

$$f(k) = \frac{1}{\sqrt{2\pi}} e^{-k^2/4\alpha} \int_{-\infty}^{\infty} dx\, e^{-\alpha\left(x+\frac{ik}{2a}\right)^2}$$
$$= \frac{1}{\sqrt{2\alpha}} e^{-k^2/4\alpha}$$

Finally

$$\tilde{\psi}(p) = \left(\frac{a^2}{\pi\hbar^2}\right)^{1/4} e^{-(p-p_0)^2 a^2/2\hbar^2}$$

which is also a Gaussian!



Schrodinger Equation for a Free Particle

Schrodinger Equation for a free particle:

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2}$$

Given $\psi(x,0)$, this equation should uniquely determine $\psi(x,t)$. Integral Solution:

$$\psi(x,t) = \int_{-\infty}^{\infty} dx' \; G(x,t;x') \; \psi(x',0)$$

where $G(x,t;x')$ satisfies Schrodinger Equation with the initial condition

$$G(x,0;x') = \delta\left(x - x'\right)$$

Because of translational invariance, it is sufficient to solve for function $G(x,t)$

$$i\hbar \frac{\partial G(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 G(x,t)}{\partial x^2}$$

with initial condition $G(x,0) = \delta(x)$. Then, $G(x,t;x',0) = G(x-x',t)$.
Interpretation of $G(x,t)$: Wavefunction of a particle at instant t which was localised at $x = 0$ at t=0 ($\psi(x,0) = \delta(x); \psi(x,t) = G(x,t)$).

Momentum Space Schrodinger Equation

Fourier transform to momentum space

$$G(x,t) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi\hbar}}\, g(p,t)\, e^{ipx/\hbar}$$

$g(p,t)$ satisfies:

$$i\hbar \frac{\partial g(p,t)}{\partial t} = \frac{p^2}{2m}\, g(p,t)$$

with solution

$$g(p,t) = g(p,0)\, e^{-i\left(p^2/2m\right)t/\hbar}$$

$G(x,0) = \delta(x) \implies g(p,0) = 1/\sqrt{2\pi\hbar}$.
Finally

$$G(x,t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{-i\left(p^2/2m\right)t/\hbar}\, e^{ipx/\hbar}$$

Note that this is the same as

$$G(x,t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{-iE_p t/\hbar}\, e^{ipx/\hbar}$$

where $E_p = p^2/2m$ is the classical expression for energy of a free particle.

$$G(x,t) = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\ e^{-i\left(p^2/2m\right)t/\hbar}\ e^{ipx/\hbar}$$

involves the Fourier transfrom of $e^{-i\left(p^2/2m\right)t/\hbar}$. We need to evaluate a transform of the form

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\ e^{-iak^2}\ e^{ikx}, \quad a > 0$$

Completing the square in the exponent, we get

$$f(x) = \frac{1}{\sqrt{2\pi}}\ e^{ix^2/4a} \int_{-\infty}^{\infty} dk\ e^{-ia(k-x/2a)^2}$$

By shifting k by $x/2a$ and scaling by $\sqrt{a}$, we get

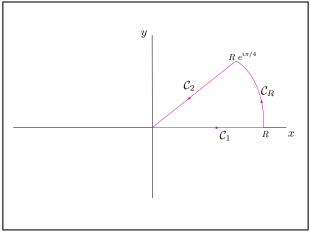
$$f(x) = \frac{1}{\sqrt{2\pi a}}\ e^{ix^2/4a} \int_{-\infty}^{\infty} dk\ e^{-ik^2}$$

We need a way to evaluate integral of the form

$$I = \int_0^{\infty} dk\ e^{ik^2} \quad \text{Fresnel Integral}$$

Fresnel Integral

Let $f(z) = e^{iz^2}$ and consider the contour integral



Since $f(z)$ is analytic everywhere,

$$\oint_{\mathcal{C}} dz\, f(z) = 0$$

Along $\mathcal{C}_1$:

$$\begin{aligned} \int_{\mathcal{C}_1} dz\, f(z) &= \int_0^R dx\, e^{ix^2} \\ &\to \int_0^\infty dx\, e^{ix^2}; \quad R \to \infty \end{aligned}$$

Along $\mathcal{C}_2$, $z = r\ e^{i\pi/4}$. Therefore $dz = dr\ e^{i\pi/4}$ with r going from R to 0. Further $z^2 = i\ r^2$. Then

$$\begin{aligned} \int_{\mathcal{C}_2} dz\ f(z) &= -e^{i\pi/4} \int_0^R dr\ e^{-r^2} \\ &\rightarrow -\frac{\sqrt{\pi}}{2}\ e^{i\pi/4}; \quad R \rightarrow \infty \end{aligned}$$

Along $\mathcal{C}_R$

$$\begin{aligned} \left| \int_{\mathcal{C}_R} dz\ e^{iz^2} \right| &\leq R \int_0^{\pi/4} d\theta\ e^{-R^2 \sin 2\theta} \\ &= \frac{R}{2} \int_0^{\pi/2} d\theta\ e^{-R^2 \sin\theta} \\ &\leq \frac{R}{2} \left(\frac{1 - e^{-R^2}}{2R^2/\pi} \right) \quad \text{(Jordan's Lemma)} \\ &\rightarrow 0 \ as\ R \rightarrow \infty \end{aligned}$$

Finally

$$\int_0^\infty dx\ e^{ix^2} = \frac{\sqrt{\pi}}{2}\ e^{i\pi/4} = \frac{\sqrt{i\pi}}{2}$$

Plugging all this back into $G(x, t)$ gives

$$G(x, t) = \sqrt{\frac{m}{2\pi i\hbar t}}\ e^{imx^2/2\hbar t}$$

This is non-zero even if $x > ct$. This is because Schrodinger Equation is not Lorentz invariant. We will (if we get time) explore this later.

Forced Harmonic Oscillator

$$\frac{d^2 x(t)}{dt^2} + \omega_0^2\, x(t) = f(t)$$

$f(t) = x(t) = 0\ \forall\ t < -T_0$. Integral solution:

$$x(t) = \int_{-\infty}^{\infty} dt'\, G(t - t') f(t')$$

where $G(t)$ (Retarded Green's Function) is given by:

$$\frac{d^2 G(t)}{dt^2} + \omega_0^2\, G(t) = \delta(t); \quad G(t) = 0 \text{ for } t < 0$$

We solve for $G(t)$ using a Fourier Integral Transform.

$$\delta(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}$$

$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, g(\omega)\, e^{i\omega t}$$

This gives

$$g(\omega) = \frac{-1}{\sqrt{2\pi}} \left(\frac{1}{\omega^2 - \omega_0^2} \right)$$

Then

$$G(t) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} d\omega\, \frac{e^{i\omega t}}{\omega^2 - \omega_0^2}$$

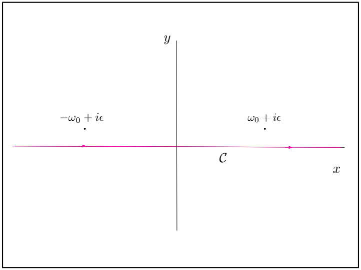
This is clearly singular! However, we still need to impose the condition that $G(t) = 0$ for $t < 0$.

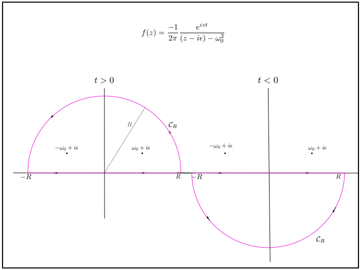
Correct prescription for $G(t)$:

$$G(t) = \frac{-1}{2\pi} \lim_{\epsilon \to 0+} \int_{-\infty}^{\infty} d\omega \, \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_0^2}$$

Visualisation in the Complex plane:

$$f(z) = \frac{-1}{2\pi} \frac{e^{izt}}{(z - i\epsilon) - \omega_0^2}$$





$$f(z) = \frac{-1}{2\pi} \frac{e^{izt}}{(z - z_1)(z - z_2)}; \quad z_1 = \omega_0 + i\epsilon, z_2 = -\omega_0 + i\epsilon$$

Using Residue Theorem,

$$G(t) = \begin{cases} \frac{\sin \omega_0 t}{\omega_0}, & t \geq 0 \\ 0, & t \leq 0 \end{cases}$$

The solution to the forced harmonic oscillator equation then becomes

$$x(t) = \int_{-\infty}^{t} dt' \, \frac{\sin\left[\omega_0(t-t')\right]}{\omega_0} \, f(t')$$

Example:

$$f(t) = \begin{cases} \sin\left(\omega_1 t\right), & |t| < \frac{N\pi}{\omega_1} \\ 0, & |t| > \frac{N\pi}{\omega_1} \end{cases}$$

Then

$$x(t) = \int_{-N\pi/\omega_1}^{t} dt' \, \frac{\sin\left[\omega_0(t-t')\right]}{\omega_0} \, f(t')$$

For $t > N\pi/\omega_1$

$$x(t) = \int_{-N\pi/\omega_1}^{N\pi/\omega_1} dt' \, \frac{\sin\left[\omega_0(t-t')\right]}{\omega_0} \, \sin\left(\omega_1 t'\right)$$

which satisfies the homogeneous equation with the solution

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

Problem

Determine A and B by expanding the sin *function in the integral.*

For $-N\pi/\omega_1 < t < N\pi/\omega_1$

$$x(t) = \int_{-N\pi/\omega_1}^{t} dt' \, \frac{\sin\left[\omega_0(t-t')\right]}{\omega_0} \, \sin\left(\omega_1 t'\right)$$

Problem

Determine $x(t)$ for $-N\pi/\omega_1 < t < N\pi/\omega_1$.

1 D Wave Equation With Source

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \phi(x,t)}{\partial t^2} = \rho(x,t); \quad \rho(x,t) = 0 \;\; \forall \; t < -T_0$$

We look for a solution such that $\phi(x,t) = 0 \; \forall \; t < -T_0$.
Green's Function:

$$\frac{\partial^2 G(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 G(x,t)}{\partial t^2} = \delta(x)\delta(t); \quad G(x,t) = 0 \; \forall \; t < 0$$

Then

$$\phi(x,t) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dt' \; G(x-x', t-t')\rho(x',t')$$

Fourier Integral transform of $G(x,t)$:

$$G(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \, g(k,\omega) \; e^{ikx} \; e^{i\omega t}$$

Expressing delta functions as Fourier Integrals:

$$\delta(x)\delta(t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ e^{ikx}\ e^{i\omega t}$$

Substitution gives

$$g(k,\omega) = \frac{v^2}{2\pi}\left(\frac{1}{\omega^2 - \omega_k^2}\right),\quad \omega_k = v\,|k|$$

The Green's function with the condition $G(x,t) = 0\ \forall\ t < 0$ is given by

$$G(x,t) = \frac{v^2}{4\pi^2} \lim_{\epsilon\to 0+} \int_{-\infty}^{\infty} dk\ e^{ikx} \int_{-\infty}^{\infty} d\omega\ \frac{e^{i\omega t}}{(\omega - i\epsilon)^2 - \omega_k^2}$$

The ω integral is the same as that for the Harmonic Oscillator. Then

$$G(x,t) = -\frac{v^2}{2\pi}\,\theta(t) \int_{-\infty}^{\infty} dk\ e^{ikx}\ \frac{\sin(\omega_k t)}{\omega_k}$$

where $\theta(t)$ is a step function

$$\theta(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dk\ e^{ikx}\ \frac{\sin(\omega_k t)}{\omega_k} \\ &= \int_{-\infty}^{\infty} dk \cos kx\ \frac{\sin(|k|\ vt)}{|k|\ v} \\ &= \int_{-\infty}^{\infty} dk \cos kx\ \frac{\sin(kvt)}{kv} \\ &= \frac{1}{2v}\left[\int_{-\infty}^{\infty} dk\ \frac{\sin k(x+vt)}{k} - \int_{-\infty}^{\infty} dk\ \frac{\sin k(x-vt)}{k}\right] \end{aligned}$$

We now need to evaluate integrals of the form

$$I(a) = \int_{-\infty}^{\infty} dx\ \frac{\sin(ax)}{x}$$

$$I(a) = \begin{cases} +I_s, & a > 0 \\ -I_s, & a < 0 \end{cases}$$

where (to be proved)

$$\begin{aligned} I_s &= \int_{-\infty}^{\infty} dx \, \frac{\sin x}{x} \\ &= \pi \end{aligned}$$

This can be written as

$$I(a) = [\theta(a) - \theta(-a)] \, \pi$$

Then

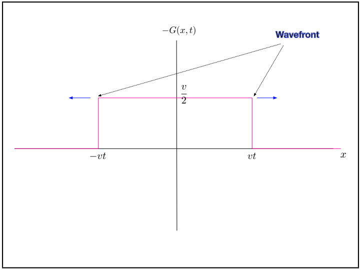
$$G(x,t) = -\pi \, \frac{v^2}{2\pi} \, \theta(t) \, \frac{1}{2v} \, [\theta(x+vt) - \theta(-x-vt) - \theta(x-vt) + \theta(-x+vt)]$$

The combination of step functions is easy to visualise

$$\theta(x+vt) - \theta(-x-vt) - \theta(x-vt) + \theta(-x+vt) = \begin{cases} 2; & |x| < vt \\ 0; & |x| > vt \end{cases}$$

Finally

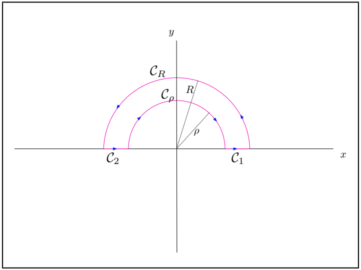
$$G(x,t) = \begin{cases} -\frac{v}{2}\,\theta(t); & |x| < vt \\ 0; & |x| > vt \end{cases}$$

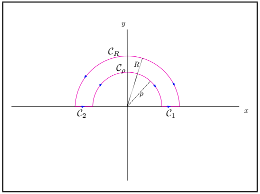


Evaluaion of

$$
\begin{aligned}
I_s &= \int_{-\infty}^{\infty} dx\, \frac{\sin x}{x} \\
&= 2\int_{0}^{\infty} dx\, \frac{\sin x}{x}
\end{aligned}
$$

Let $f(z) = e^{iz}/z$. We consider the closed contour integral of $f(z)$ around the following contour





Since $f(z)$ is analytic all along and within the contour, it follows that

$$\begin{aligned} \int_{\mathcal{C}_1} dz\, f(z) + \int_{\mathcal{C}_2} dz\, f(z) + \int_{\mathcal{C}_\rho} dz\, f(z) + \int_{\mathcal{C}_R} dz\, f(z) &= 0 \\ \Longrightarrow \int_{\mathcal{C}_1} dz\, f(z) + \int_{\mathcal{C}_2} dz\, f(z) &= -\int_{\mathcal{C}_\rho} dz\, f(z) - \int_{\mathcal{C}_R} dz\, f(z) \end{aligned}$$

Along $\mathcal{C}_1,\ z = r$ and along $\mathcal{C}_2,\ z = -r$. Therefore

$$\int_{\mathcal{C}_1} dz\, f(z) + \int_{\mathcal{C}_2} dz\, f(z) = 2i \int_\rho^R dr\, \frac{\sin r}{r}$$

Then

$$2i \int_{\rho}^{R} dr \, \frac{\sin r}{r} = -\int_{\mathcal{C}_\rho} dz \, \frac{e^{iz}}{z} - \int_{\mathcal{C}_R} dz \, \frac{e^{iz}}{z}$$

We now take the limits $\rho \to 0$ and $R \to \infty$. From Jordan's Lemma, the integral over $\mathcal{C}_R$ goes to zero as $R \to \infty$. We need to evaluate the integral over $\mathcal{C}_\rho$ in the limit $\rho \to 0$. Along $\mathcal{C}_\rho$, $z = \rho \, e^{i\theta}$, $\theta \in [0, \pi]$. Then

$$\begin{aligned}
\int_{\mathcal{C}_\rho} dz \, \frac{e^{iz}}{z} &= i\rho \int_{\pi}^{0} d\theta \, e^{i\theta} \, \frac{e^{i\rho(\cos\theta + i\sin\theta)}}{\rho \, e^{i\theta}} \\
&= -i \int_{0}^{\pi} d\theta \, e^{i\rho(\cos\theta + i\sin\theta)}
\end{aligned}$$

In the limit $\rho \to 0$, this is $-i\pi$. Then

$$\int_{0}^{\infty} dr \, \frac{\sin r}{r} = \frac{\pi}{2}$$

which gives $I_s = \pi$.

Homogeneous Wave Equation

We now solve the equation

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \phi(x,t)}{\partial t^2} = 0$$

with initial condition $\phi(x,0) = \phi_0(x)$ and $\partial\phi/\partial t|_{t=0} = 0$. We take the Fourier Integral Transform of $\phi(x,t)$

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \tilde{\phi}(k,t)\, e^{ikx}$$

Substitution gives

$$\frac{\partial^2 \tilde{\phi}(k,t)}{\partial t^2} = -k^2 v^2\, \tilde{\phi}(k,t)$$

the general solution to which is

$$\tilde{\phi}(k,t) = A_k \cos(kvt) + B_k \sin(kvt)$$

The initial condition gives $B_k = 0$ and $A_k = \tilde{\phi}_0(k)$, the Fourier Transform of $\phi_0(x)$.

Then

$$
\begin{aligned}
\phi(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \tilde{\phi}_0(k) \cos(kvt)\, e^{ikx} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi}}\, \phi_0(x') \left(e^{ikvt} + e^{-ikvt}\right)\, e^{-ikx'}\, e^{ikx} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} dx'\, \phi_0(x') \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(e^{ik(x+vt-x')} + e^{ik(x-vt-x')}\right) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} dx'\, \phi_0(x')\, \left(\delta(x+vt-x') + \delta(x-vt-x')\right) \\
&= \frac{1}{2}\left[\phi_0(x+vt) + \phi_0(x-vt)\right]
\end{aligned}
$$

Problem

Using Fourier Integral Transform, solve the homogeneous wave equation with the initial condition $\phi(x,0) = 0$ *and* $\partial\phi/\partial t|_{t=0} = v_0(x)$.

Diffusion Equation

$$\frac{\partial \phi(\vec{x}, t)}{\partial t} = D\, \nabla^2 \phi(\vec{x}, t)$$

3-D Fourier Transform:

$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}}\, \tilde{\phi}(\vec{k}, t)\, e^{i\vec{k}\cdot\vec{x}}$$

where $d^3k = dk_x dk_y dk_z$ and $\vec{k} \cdot \vec{x} = k_x x + k_y y + k_z z$. Substitution gives

$$\frac{\partial \tilde{\phi}(\vec{k}, t)}{\partial t} = -D\, k^2\, \tilde{\phi}(\vec{k}, t)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. the solution is

$$\tilde{\phi}(\vec{k}, t) = \tilde{\phi}(\vec{k}, 0) e^{-Dk^2 t}$$

The Fourier Integral reduces to

$$\phi(\vec{x}, t) \quad = \quad \int \frac{d^3k}{(2\pi)^{3/2}}\, \tilde{\phi}(\vec{k}, 0)\, e^{-Dk^2 t}\, e^{i\vec{k}\cdot\vec{x}}$$

Using the inverse transform at $t = 0$

$$\tilde{\phi}(\vec{k}, 0) = \int \frac{d^3x'}{(2\pi)^{3/2}} \; \phi(\vec{x'}, 0) \; e^{-i\vec{k}\cdot\vec{x'}}$$

we get

$$\phi(\vec{x}, t) = \int d^3x' \; G(\vec{x}, t; \vec{x'}) \; \phi(\vec{x'}, 0)$$

where

$$G(\vec{x}, t; \vec{x'}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{-Dk^2t} \; e^{i\vec{k}\cdot(\vec{x}-\vec{x'})}$$

It is easy to check that $G(\vec{x}, t; \vec{x'})$ is just the time evolved density distribution if the density at $t = 0$ was $\phi(\vec{x}, 0) = \delta^3(\vec{x} - \vec{x'})$. This is just the Green's function. It can be evaluated as

$$G(\vec{x}, t; \vec{x'}) = I(x - x', t) \; I(y - y', t) \; I(z - z', t)$$

where

$$I(a) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} e^{-Dp^2t} \; e^{ipa}$$

This is just the Fourier transfrom of a Gaussian function with the result

$$I(a) = \frac{1}{\sqrt{2Dt}}\ e^{-a^2/4Dt}$$

Finally,

$$G(\vec{x}, t; \vec{x'}) = \frac{1}{(2Dt)^{3/2}}\ e^{-(\vec{x}-\vec{x'})^2/4Dt}$$

This solves the diffusion equation.