Integral Transforms

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Outline

Linear Vector Spaces

Let $\mathbb{F}$ be the set of real ($\mathbb{R}$) or complex ($\mathbb{C}$) numbers. A set of objects V is called a 'vector space over $\mathbb{F}$' if $\forall \alpha, \beta, \gamma \in V$ (called vectors) and $\forall a, b \in \mathbb{F}$ (called scalars), the following hold (with an operation of 'addition' of vectors and 'multiplication' of a vector by a scalar defined)

1. $\alpha + \beta \in V$ (Closure)
2. $\alpha + \beta = \beta + \alpha$ (Addition is Commutative)
3. $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ (Addition is Associative)
4. Existence of a *null* or 'zero' vector $\phi \in V$: $\alpha + \phi = \phi + \alpha = \alpha$
5. Existence of additive inverse: For each $\alpha \in V$, $\exists \, (-\alpha) \in V$: $\alpha + (-\alpha) = \phi$
6. $a\, \alpha \in V$
7. $a\, (\alpha + \beta) = a\, \alpha + a\, \beta$
8. $(a + b)\, \alpha = a\, \alpha + b\, \alpha$
9. $a\, (b\, \alpha) = (ab)\, \alpha$
10. $1.\alpha = \alpha$

Examples

Examples:

- Set of all directed arrows in a plane, with addition defined by the parallelogram law and multipliation with a real number defined as scaling the length of the arrow.
- Set of all continuous functions of a real variable defined over some interval.
- Solutions to homogeneous linear differential equations.
- Set of n-tuples $(x_1, x_2, .., x_n)$ of real or complex numbers with addition and multiplication by a number defined intuitively. This set is called $\mathbb{R}^n$ if the numbers are real, and $\mathbb{C}^n$ if they are complex.
- Set of all 'square-integrable' complex functions of a real variable over an interval (a, b)

$$\int_a^b dx\ |\psi(x)|^2 < \infty \tag{1}$$

  This set is called $L^2(a, b)$.
- Infinite set of complex numbers $\{x_k\}$; $k = 1, 2, 3, ...$ with the condition

$$\sum_{k=1}^{\infty} {x_k}^2 < \infty \tag{2}$$

  This set is called l^2.

Linear Independence

Definition

A set of vectors $\alpha_1, \alpha_2, .., \alpha_n$ is said to be *linearly independent* (LI) if an equation of the form

$$c_1\ \alpha_1 + c_2\ \alpha_2 + .. + c_n\ \alpha_n = \phi$$

has a unique solution $c_1 = c_2 = .. = c_n = 0$. An infinite set of vectors is said to be linearly independent if any finite subset of the set of vectors is linearly independent.

Definition

If a set of vectors is not linearly independent, it is said to be *linearly dependent*.

If a set of vectors is linearly dependent then at least one of them can be expressed as a linear combintion of the others. Say, the set $\alpha_1, \alpha_2, .., \alpha_n$ is linearly dependent. Consider the equation

$$c_1\ \alpha_1 + c_2\ \alpha_2 + .. + c_n\ \alpha_n = \phi$$

Since the set is not linearly independent, at least two of the coefficients are non-zero. Say, $c_1 \neq 0$. Then

$$\alpha_1 = -\frac{c_2}{c_1}\ \alpha_2 - \frac{c_3}{c_1}\ \alpha_3 .. - \frac{c_n}{c_1}\ \alpha_n$$

Example

Linear Independence of functions $f_1(x), f_2(x), .., f_n(x) \in L^2(a, b)$.

$$c_1 f_1(x) + c_2 f_2(x) + .. + c_n f_n(x) = 0$$

Differentiating $n - 1$ times

$$\begin{aligned}
c_1 f_1(x) + c_2 f_2(x) + .. + c_n f_n(x) &= 0 \\
c_1 f_1^{(1)}(x) + c_2 f_2^{(1)}(x) + .. + c_n f_n^{(1)}(x) &= 0 \\
c_1 f_1^{(2)}(x) + c_2 f_2^{(2)}(x) + .. + c_n f_n^{(2)}(x) &= 0 \\
\dots\dots\dots\dots &= .. \\
c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + .. + c_n f_n^{(n-1)}(x) &= 0
\end{aligned}$$

Matrix Equation

$$\begin{pmatrix} f_1(x) & f_2(x) & . & f_n(x) \\ f_1^{(1)}(x) & f_2^{(1)}(x) & . & f_n^{(1)}(x) \\ . & . & . & . \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & . & f_n^{(n-1)}(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ . \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \end{pmatrix}$$

A sufficient condition for a set of functions to be LI is that their *Wronskian* (detrminant of the matrix) is non-vanishing at one or more points in the interval.

Basis

Definition

A maximal LI subset of a vector space V is called a *basis* of V.

Clearly, a basis spans the vector space.

Theorem

Every vector of a vector space has a unique representation as a linear combination of the vectors of a basis of the vector space.

Theorem

Every basis of a vector space has the same number of elements.

Examples

Example

Consider the vector space of all polynomials of degree n. The set $\{1, x, x^2, .., x^n\}$ is a basis for this space.

Example

Consider the vector space $L^2(0, 1)$. The infinite set $\{1, \cos(2\pi kx), \sin(2\pi kx)\}$ for $k = 1, 2, .., ..$ is LI (since any finite subset is LI). From Fourier's Theorem, any $f(x) \in L^2(0, 1)$ can be expanded as a series of these functions. Then, this set spans $L^2(0, 1)$. The vector space $L^2(0, 1)$ is then *infinite dimensional*.

Inner Product

Definition

An *inner product* on a vector space V over a field $\mathbb{F}$ is an assignment to each pair of vectors α and β a scalar $(\alpha|\beta) \in \mathbb{F}$, satisfying the following conditions

1. $(\alpha|a\beta + b\gamma) = a(\alpha|\beta) + b(\alpha|\gamma) \quad \forall a, b \in \mathbb{F}$ (Linearity).
2. $(\alpha|\beta)^* = (\beta|\alpha)$.
3. $(\alpha|\alpha) \geq 0$ and $(\alpha|\alpha) = 0 \Leftrightarrow \alpha = \phi$.

Note that the order in which the vectors appear in the inner product is important.

Example

Inner product on $L^2(a, b)$

$$(f|g) = \int_a^b dx\ f(x)^* g(x)$$

Generalisation

$$(f|g) = \int_a^b dx\ f(x)^* g(x) w(x)$$

$w(x) \geq 0$: *weight function.*

Schwarz Inequality

Theorem

Schwarz Inequality: Any inner product on a complex vector space satisfies the inequality

$$|(\alpha|\beta)| \leq \sqrt{(\alpha|\alpha)}\,\sqrt{(\beta|\beta)}$$

Proof.

Let $\psi = \alpha + c\beta$. Then, since $(\psi|\psi) \geq 0$,

$$\begin{aligned} (\alpha + c\beta|\alpha + c\beta) &\geq 0 \\ \Rightarrow (\alpha|\alpha) + c^* c(\beta|\beta) + c(\alpha|\beta) + c^*(\beta|\alpha) &\geq 0 \end{aligned}$$

Choosing $c = -(\beta|\alpha)/(\beta|\beta)$ gives the inequality. □

Note: The equality holds only if $(\psi|\psi) = 0$, which implies that $\psi = \alpha + c\beta = 0$. That is, α and β are linearly dependent.

Norm

Definition

Given an innerproduct defined on a vector space, a *norm* (length) can be defined as

$$\| \alpha \| = \sqrt{(\alpha|\alpha)}$$

The norm satisfies the ‘Triangle Inequality’:

$$\| \alpha + \beta \| \leq \| \alpha \| + \| \beta \|$$

Definition

A pair of vectors is said to be *orthogonal* if their inner product vanishes.

Definition

A vector is said to be *normalized* if its norm is equal to unity.

Definition

A pair of vectors α and β is said to be an *orthonormal pair* if $(\alpha|\beta) = 0$ and $\| \alpha \| = \| \beta \| = 1$.

Definition

In a finite dimensional vector space, an *orthonormal basis* is a basis set $\alpha_1, \alpha_2, .., \alpha_n$ (where n is the dimension of the space) such that

$$(\alpha_i|\alpha_j) = \delta_{ij}$$

Example

In the space $L^2(0, 1)$, the set of functions $\psi_n(x) = \sqrt{2}\cos(2\pi nx)$ for $n = 0, 1, ..$ and $\psi_n(x) = \sqrt{2}\sin(2\pi nx)$ for $n = 1, 2, ...$ form an infinite orthonormal set. In fact, Fourier's theorem tells us that they form a basis for $L^2(0, 1)$.

Infinite Dimensional Vector Spaces

Let $\{\alpha_i\}$ be an orthonormal basis in an infinite dimensional space V. How do we interpret

$$u = \sum_{i=1}^{\infty} c_i \, \alpha_i$$

Definition

Let V be an inner product space with a norm $\| \cdot \|$. A sequence $\{\alpha_1, \alpha_2, ..\alpha_n\}$ of vectors in V converges to $\alpha \in V$ if

$$\| \alpha - \alpha_n \| \to 0 \text{ as } n \to \infty$$

Partial sum

$$u_n = \sum_{i=1}^{n} c_i \, \alpha_i$$

Then, $u = \lim_{n\to\infty} u_n \leftrightarrow \| u - u_n \| \to 0$ *as* $n \to \infty$.

Theorem

Given

$$v = \sum_{i=1}^{n} c_i \, \alpha_i,$$

1. $c_i = (\alpha_i | v)$, *so that*

$$v = \sum_{i=1}^{n} \alpha_i \, (\alpha_i | v)$$

2. $$\| \, v \, \|^2 = \sum_{i=1}^{n} |(\alpha_i | v)|^2$$

Proof: Take inner product of both sides with α_i and use linearity and orthonormality.

Bessel's Inequality

Theorem

Bessel's Inequality

1. $\sum_{i=1}^{\infty} |(\alpha_i|u)|^2$ *converges.*
2. $\sum_{i=1}^{\infty} |(\alpha_i|u)|^2 \leq \| u \|^2$

Definition

Completeness: An orthonormal set $\{\alpha_1, \alpha_2, ..\alpha_n, ...\}$ is said to be *complete* if $\forall u \in V$

$$u = \sum_{i=1}^{\infty} \alpha_i \, (\alpha_i|u)$$

Theorem

An orthonormal set $\{\alpha_1, \alpha_2, ..\alpha_n, ...\}$ *is complete iff*

$$\| u \|^2 = \sum_{i=1}^{\infty} |(\alpha_i|u)|^2$$

Theorem

In the space $L^2(0,1)$*, the set of orthonormal functions* $\psi_n(x) = \sqrt{2}\cos(2\pi nx)$ *for* $n = 0, 1, ..$ *and* $\psi_n(x) = \sqrt{2}\sin(2\pi nx)$ *for* $n = 1, 2, ...$ *form a complete set.*

Space $L^2(-a, a)$

Inner Product:

$$(f|g) = \int_{-a}^{a} dx\, f^*(x) g(x)$$

Orthonormal Basis: The set

$$\{u_n\} = \left\{1/\sqrt{2a},\ (1/\sqrt{a}) \cos(k_n x),\ (1/\sqrt{a}) \sin(k_n x)\right\};\ n = 1, 2, 3...$$

where

$$k_n = \frac{n\,\pi}{a}$$

Equivalently, the complex set

$$\{\phi_n\} = \left\{\left(1/\sqrt{2a}\right) e^{ik_n x}\right\};\ \ n = 0, \pm 1, \pm 2, \pm 3, ...$$

forms an orthonormal basis.

Then, $\forall\ f(x) \in L^2(-a, a)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, \phi_n(x) = \frac{1}{\sqrt{2a}} \sum_{n=-\infty}^{\infty} c_n\, e^{ik_n x}$$

where

$$c_n = (\phi_n|f) = \frac{1}{\sqrt{2a}} \int_{-a}^{a} dx\, f(x)\, e^{-ik_n x}$$

Transition to Fourier Integral

We take the limit $a \to \infty$ to go from $L^2(-a, a)$ to $L^2(\mathbb{R})$. In this limit, $\Delta k_n = k_{n+1} - k_n = \pi/a \to dk$. Further,

$$\begin{aligned} \sum_n g(k_n) &= \sum_n \Delta n \, g(k_n) \\ &= \frac{a}{\pi} \sum_{k_n} \Delta k_n \, g(k_n) \\ &\to \frac{a}{\pi} \int_{-\infty}^{\infty} dk \; g(k) \end{aligned}$$

Define $\tilde{f}(k_n) = \sqrt{a/\pi}\; c_n$.

Then, in the limit $a \to \infty$,

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2a}} \sum_{n=-\infty}^{\infty} c_n \, e^{ik_n x} \\ &= \frac{1}{\sqrt{2a}} \times \frac{a}{\pi} \times \sqrt{\frac{\pi}{a}} \sum_{k_n} \Delta k_n \, \tilde{f}(k_n) \;\; e^{ik_n x} \\ &\to \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \, \tilde{f}(k) \, e^{ikx} \end{aligned}$$

where

$$\begin{aligned} \tilde{f}(k) &= \lim_{a\to\infty} \tilde{f}\,(k_n) \\ &= \lim_{a\to\infty} \sqrt{\frac{a}{\pi}} \times \frac{1}{\sqrt{2a}} \int_{-a}^{a} dx \, f(x) \, e^{-ik_n x} \\ &= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \, f(x) \, e^{-ikx} \end{aligned}$$

Fourier Integral Transform

Fourier Integral Transform:

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \tilde{f}(k)\, e^{ikx}$$

Inverse Transform:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, f(x)\, e^{-ikx}$$

Quantum Mechanics: Position vs Momentum Space