Vector Fields

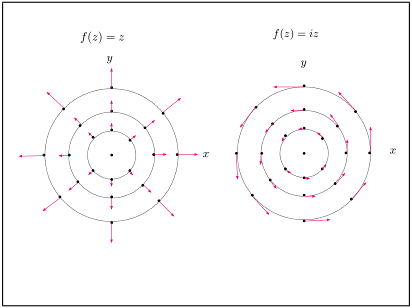
A. Gupta

[1] Department of Physics
St. Stephen's College

Outline

Complex Functions as Vector Fields

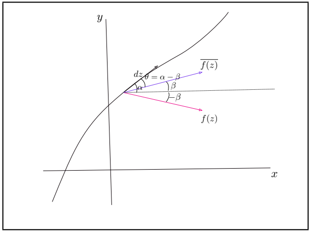
Prescription: Attach a vector $f(z)$ with tail at point z.



Problem: $\int_C dz\; f(z)$ has no simple interpretation in temrs of line and ‘surface’ integrals.

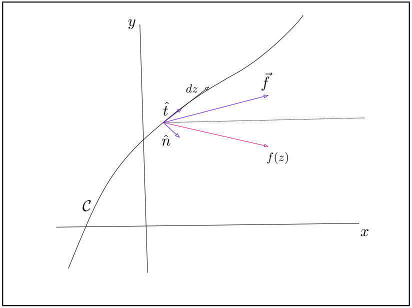
The Polya Field

Resolution: Visualize $\overline{f(z)}$ as a vector field (Polya Vector Field).

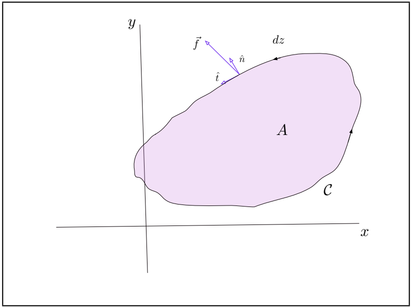


$$
\begin{aligned}
dz\ f(z) &= |f|\, e^{-i\beta} ds e^{i\alpha} \\
&= |f|\, ds e^{i\theta}; \quad \theta = \alpha - \beta \\
&= \left|\bar{f}\right| (\cos\theta + i \sin\theta)\, ds \\
&= \left|\bar{f}\right| ds \cos\theta + i \left|\bar{f}\right| ds \sin\theta \\
&= \left(\vec{f} \cdot \hat{t}\right) ds + i \left(\vec{f} \cdot \hat{n}\right) ds
\end{aligned}
$$

where $\vec{f}$ is the Polya Vector, $\hat{t}$ is unit tangent and $\hat{n}$ is unit normal to the curve.



$$\int_{\mathcal{C}} dz\, f(z) = \int_{\mathcal{C}} ds\, \vec{f} \cdot \hat{t} + i \int_{\mathcal{C}} ds\, \vec{f} \cdot \hat{n}$$
$$= \mathcal{W} + \mathcal{F}$$



$$
\begin{aligned}
\oint_{\mathcal{C}} dz\, f(z) &= \oint_{\mathcal{C}} ds\, \vec{f}\cdot\hat{t} \;+\; i\oint_{\mathcal{C}} ds\, \vec{f}\cdot\hat{n} \\
&= \iint_A dA\left(\vec{\nabla}\times\vec{f}\right) + i\iint_A dA\left(\vec{\nabla}\cdot\vec{f}\right)
\end{aligned}
$$

where $\vec{\nabla}\cdot\vec{f} = \partial_x f_x + \partial_y f_y$ and $\vec{\nabla}\times\vec{f} = \partial_x f_y - \partial_y f_x$.

Analyticity and Polya Field

The integral result demonstrates that

Analyticity and Polya Field

The Polya Field associated with an analytic function has zero divergence and curl in the region of analyticity.

Problem

Verify the above using Cauchy-Riemann Equations.

Example

$$\oint_{\mathcal{C}} dz\ \bar{z} = 2i\ A$$

Polya Field: $\vec{f} = \vec{r}$, with $\vec{\nabla} \cdot \vec{f} = 2$ and $\vec{\nabla} \times \vec{f} = 0$. Therefore

$$\oint_{\mathcal{C}} dz\ \bar{z} = 2i \iint_A dA = 2i\ A$$

Example

$$\oint_{\mathcal{C}} \frac{dz}{z} = 2\pi i$$

Polya Field: $\vec{f} = \vec{r}/r^2$ with $\vec{\nabla} \cdot \vec{f} = 2\pi\delta^{(2)}(\vec{r})$ and $\vec{\nabla} \times \vec{f} = 0$. Then

$$\oint_{\mathcal{C}} \frac{dz}{z} = 2\pi i \iint_A dA\ \delta^{(2)}(\vec{r}) = 2\pi i$$

This is physically the electric field due to a point charge (actually, a line charge).

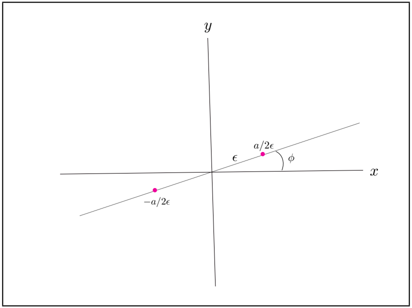
Problem

Analyze the Polya field associated with $f(z) = i/z$.

Dipole Field

Polya field of $f(z) = A/z^2$ where $A = a\,e^{i\phi}$ is complex.

$$f(z) \quad = \quad \lim_{\epsilon \to 0} \left[\frac{a/2\epsilon}{z - \epsilon\, e^{i\phi}} - \frac{a/2\epsilon}{z + \epsilon\, e^{i\phi}} \right]$$



This is just a dipole field.

Multipole Fields

Similarly, $f(z) = A/z^3$ represents a quadruploe field, etc. A Laurent series is then a multiplole expansion.