Complex Integration

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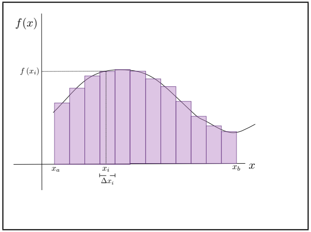
Outline

Riemann Integral

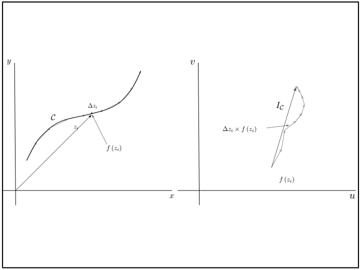
Integral of a real function $f(x)$

$$I_R = \int_{x_a}^{x_b} dx\ f(x) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\, \Delta x_i$$



Complex Integral

$$I_C = \int_{\mathcal{C}} dz\, f(x) = \lim_{n \to \infty} \sum_{i=1}^{n} f(z_i)\, \Delta z_i$$



Integral Identities

Identities

$$\left|\int_{\mathcal{C}} dz\, f(z)\right| \leq |f(z)|_{\text{max}}\; l_{\mathcal{C}}$$

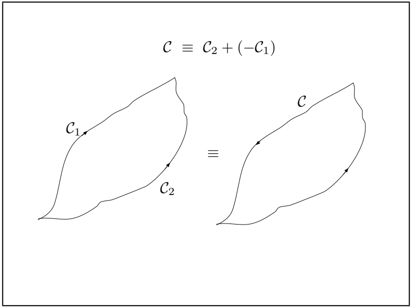
$$\int_{\mathcal{C}} dz\, a\, f(z) = a \int_{\mathcal{C}} dz\, f(z)$$

$$\int_{\mathcal{C}} dz\, [f(z) + g(z)] = \int_{\mathcal{C}} dz\, f(z) + \int_{\mathcal{C}} dz\, g(z)$$

$$\int_{\mathcal{C}_1 + \mathcal{C}_2} dz\, f(z) = \int_{\mathcal{C}_1} dz\, f(z) + \int_{\mathcal{C}_2} dz\, f(z)$$

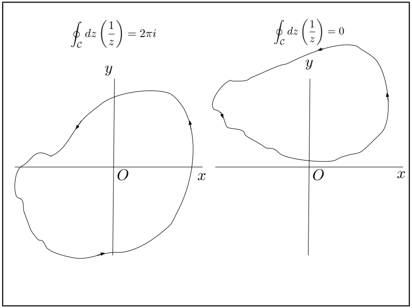
$$\int_{-\mathcal{C}} dz\, f(z) = -\int_{\mathcal{C}} dz\, f(z)$$

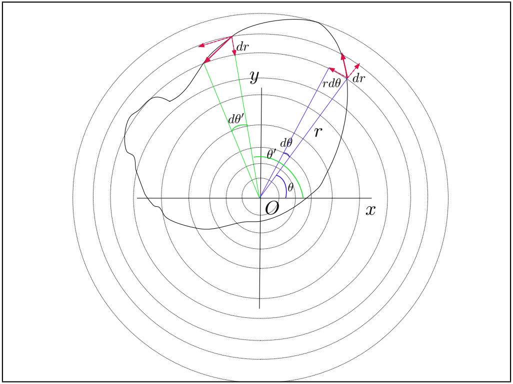
$$\int_{\mathcal{C}_1} dz\, f(z) - \int_{\mathcal{C}_2} dz\, f(z) = \oint_{\mathcal{C}} dz\, f(z)$$



Complex Inversion

$$\oint_C dz \left(\frac{1}{z}\right)$$




dr
y
$r\,d\theta$
dr
$d\theta'$
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θ
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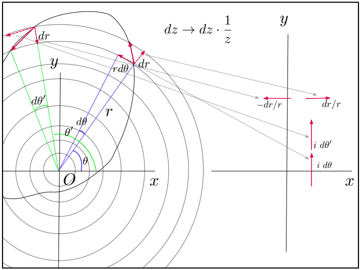
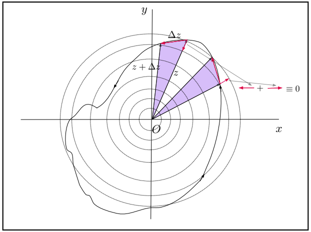


$$\oint_{\mathcal{C}} dz \left(\frac{1}{z}\right) = i \sum d\theta$$

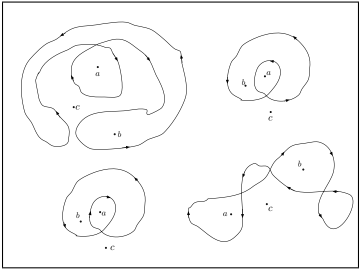
$$= \begin{cases} 2\pi i & \text{if } \mathcal{C} \text{ encloses origin} \\ 0 & \text{if } \mathcal{C} \text{ does not enclose origin} \end{cases}$$

$$I_{\mathcal{C}} = \oint_C dz\, \bar{z}$$



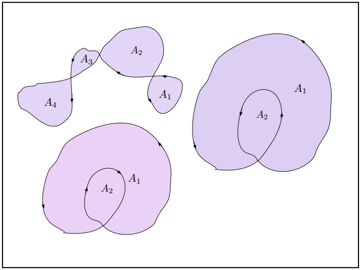
Area of triangle: $\frac{1}{2}\,\mathrm{Im}\,[(z+\Delta z)\,\bar{z}] = \frac{1}{2}\,\mathrm{Im}\,(\Delta z\,\bar{z})$.
Also, $\mathrm{Re}\,(I_{\mathcal{C}}) = 0$.

$$\Longrightarrow \oint_C dz\,\bar{z} = 2i\,(\text{Area enclosed by } \mathcal{C})$$



$$\oint_{\mathcal{C}} dz \left(\frac{1}{z - z_0} \right) = 2\pi i \, \nu \left(z_0 \right)$$

where $\nu \left(z_0 \right)$ is the 'winding number' of loop $\mathcal{C}$ about point z_0.



$$\oint_C dz\, \bar{z} = 2i \sum_j \nu_j\, A_j$$

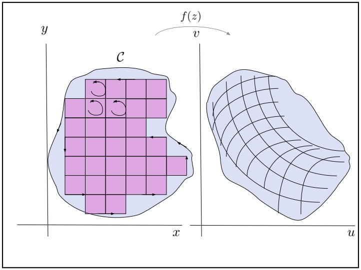
Cauchy's Theorem

Theorem

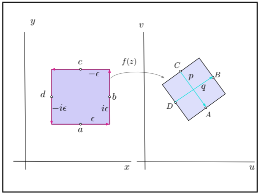
Cauchy's Theorem
If a function $f(z)$ is analytic at all points interior to and on a simple closed contour $\mathcal{C}$, then

$$\oint_C dz\, f(z) = 0$$

Proof



$$\oint_C dz\ f(z) = \sum_i \oint_{\Box_i} dz\ f(z)$$



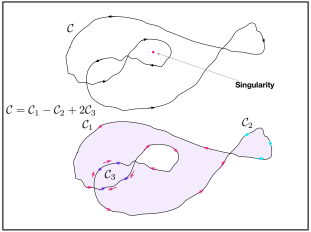
$$\sum_i \oint_{\square_i} dz\, f(z) = A(\epsilon) + C(-\epsilon) + B(i\epsilon) + D(-i\epsilon)$$
$$= p\,\epsilon + iq\,\epsilon$$
$$= 0$$

since $iq = -p$ (small square is mapped to a square).

Generalization of Cauchy's Theorem

Theorem

If an analytic function is such that it has no singularities inside a closed loop (the winding number around the singularity is zero) then its integral around the loop is zero.



Deformation Principle

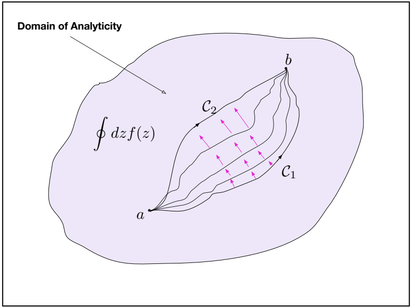
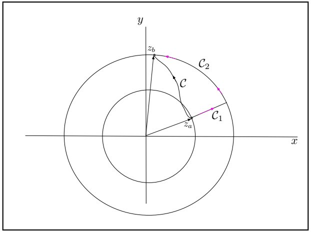


Illustration: $\int_{\mathcal{C}} dz\, z^n$



$$\int_{\mathcal{C}} dz\, z^n = \int_{\mathcal{C}_1} dz\, z^n + \int_{\mathcal{C}_2} dz\, z^n$$

Parametric Integration

To evaluate $\int_{z_a}^{z_b} dz\ f(z)$, choose a parameter $t \in \mathbb{R}$ along the curve such that $z = z(t)$ with $z(t_a) = z_a, z(t_b) = z_b$. Then,

$$\int_{z_a}^{z_b} dz\ f(z) = \int_{t_a}^{t_b} \frac{dz}{dt}\ f\left(z(t)\right) dt$$

Along $\mathcal{C}_1$, $z = r\ e^{i\theta_a},\ \ r \in [r_a, r_b]$ and $dz = e^{i\theta_a}\ dr$. Then,

$$\begin{aligned}\int_{\mathcal{C}_1} dz\ z^n &= e^{i(n+1)\theta_a} \int_{r_a}^{r_b} dr\ r^n \\ &= e^{i(n+1)\theta_a} \left(\frac{r_b^{n+1} - r_a^{n+1}}{n+1} \right)\end{aligned}$$

Along $\mathcal{C}_2$, $z = r_b\, e^{i\theta} = r_b\,(\cos\theta + i\sin\theta)\,,\ \theta \in [\theta_a, \theta_b]$ and $dz = r_b\,(-\sin\theta + i\cos\theta)\, d\theta = i\, r_b\, e^{i\theta}\, d\theta$. Then,

$$\begin{aligned}
\int_{\mathcal{C}_2} dz\, z^n &= i\, r_b^{n+1} \int_{\theta_a}^{\theta_b} d\theta\, e^{i(n+1)\theta} \\
&= i\, r_b^{n+1} \int_{\theta_a}^{\theta_b} d\theta\, (\cos(n+1)\theta + i\sin(n+1)\theta) \\
&= \frac{r_b^{n+1}}{n+1} \left(e^{(n+1)\theta_b} - e^{(n+1)\theta_a} \right)
\end{aligned}$$

Finally,

$$\begin{aligned}\int_{\mathcal{C}} dz\, z^n &= \int_{\mathcal{C}_1} dz\, z^n + \int_{\mathcal{C}_2} dz\, z^n \\ &= \frac{r_b^{n+1}\, e^{i(n+1)\theta_b} - r_a^{n+1}\, e^{i(n+1)\theta_a}}{n+1} \\ &= \frac{z_b^{n+1} - z_a^{n+1}}{n+1}\end{aligned}$$

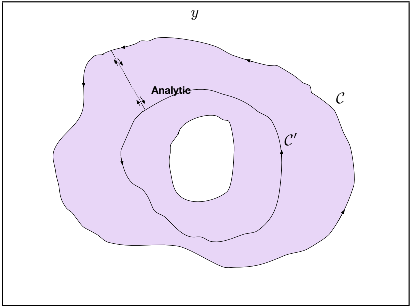
which is formally similar to the real result

$$\int_{x_a}^{x_b} dx\, x^n = \frac{x_b^{n+1} - x_a^{n+1}}{n+1}$$

Problem

Using a suitable contour, show that $\int_{z_a}^{z_b} dz\, e^z = e^{z_b} - e^{z_a}$.

More Deformation



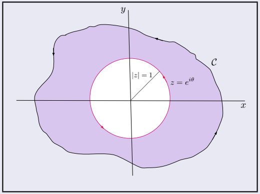
$$\oint_{\mathcal{C}} dz\, f(z) = \oint_{\mathcal{C}'} dz\, f(z)$$

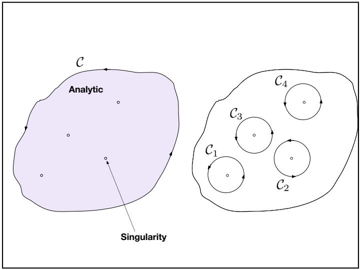
Problem

Show that

$$\oint_{\mathcal{C}} dz\, \frac{1}{z^n} = 0; \quad n = 2, 3, 4, \ldots$$

where $\mathcal{C}$ is a closed contour circling the origin.
Hint:

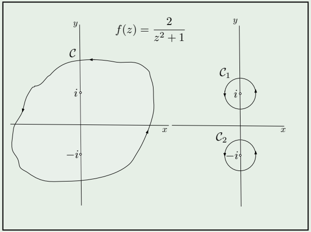




$$\oint_{\mathcal{C}} dz\ f(z) = \sum_i \oint_{\mathcal{C}_i} dz\ f(z)$$

Example

$f(z) = 2/\left(z^2+1\right) = i/(z+i) - i/(z-i)$



$$\oint_{\mathcal{C}} dz\, f(z) = 0$$

Problem

Evaluate the integral of $f(z) = z^5/(z+1)^2$ *around a simple closed contour circling* $z = -1$ *by expressing* $f(z)$ *in terms of partial fractions.*

Problem

Evaluate the integral of $f(z) = \sin z/(z+1)^6$ *around a simple closed contour circling the origin by expressing* $\sin z$ *in its series form and integrating term by term.*

Antiderivatives

Theorem

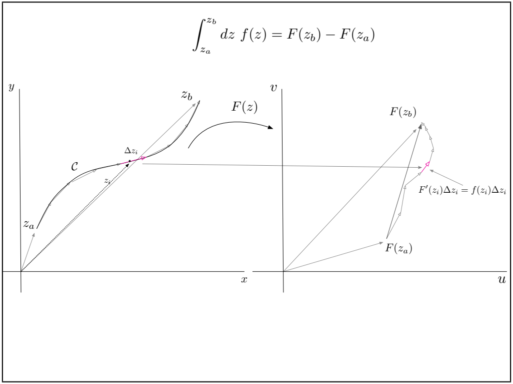
Let $f(z)$ be analytic over a domain $\mathcal{D}$. Then there exists an analytic function $F(z)$ such that $F(z) = f'(z)$ and

$$\int_{z_a}^{z_b} dz\, f(z) = F(z_b) - F(z_a)$$

*$F(z)$ is called the **antiderivative** of $f(z)$.*

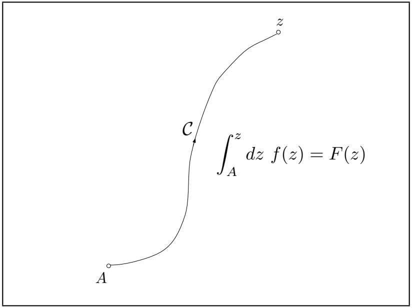
Clearly, if $F(z)$ is an antiderivative, so is $\tilde{F}(z) = F(z) + c$ where c is a complex number.

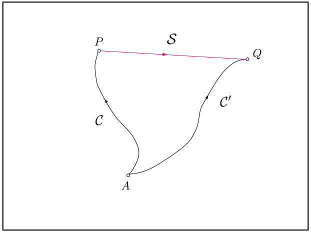
If $F(z)$ exists, it is clear that $\int_{z_a}^{z_b} dz\, f(z) = F(z_b) - F(z_a)$.



Existence and analyticity of $F(z)$: Path independence of integral of $f(z)$ allows us to define

$$F(z) = \int_A^z dz\, f(z)$$





$$F(Q) - F(P) = \int_{\mathcal{S}} dz\, f(z)$$

Let $\Delta = P - Q$ be infinitesimal. This implies Δ is mapped by $F(z)$ to $\int_{\mathcal{S}} dz\, f(z) \approx f(P) \times \Delta$. Therefore $F(z)$ is a map that scales and rotates by a fixed amount at any point.

$$\implies F(z) \text{ is analytic}$$

Cauchy's Integral Formula

Let $f(z)$ be analytic on and within a simple closed contour $\mathcal{C}$ enclosing point a. Then

$$\frac{1}{2\pi i}\oint_{\mathcal{C}} dz \frac{f(z)}{z-a} = f(a)$$

Proof:
Deforming the contour to a circle of radius R about $z = a$ and writing $z = a + Re^{i\theta}$ (so that $dz = iRe^{i\theta}$)

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\mathcal{C}} dz \frac{f(z)}{z-a} &= \frac{1}{2\pi}\int_0^{2\pi} d\theta\, f\left(a + Re^{i\theta}\right) \\ &\to f(a) \text{ as } R \to 0\end{aligned}$$

Infinite Differentiability

Theorem

An analytic function $f(z)$ is infinitely differentiable within the domain of analyticity $\mathcal{D}$ and its n^{th} derivative at a point $a \in \mathcal{D}$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C dz \, \frac{f(z)}{(z-a)^{n+1}}$$

Proof:
Define

$$F_a(z) = \frac{f(z) - f(a)}{z - a}$$

Let $z = a + \xi$ be an infinitesimal vector from point a. Then, in the limit $\xi \to 0$

$$\begin{aligned} F_a(a+\xi) &= \frac{f(a+\xi) - f(a)}{\xi} \\ &\to \frac{f'(a)\xi}{\xi} \\ &= f'(a) \end{aligned}$$

$\therefore$ $F_a(a)$ exists and is finite. Since $F_a(z)$ is analytic everywhere else in $\mathcal{D}$, it is analytic everywhre in $\mathcal{D}$.

We can use Cauchy's Integral Formula on $F_a(z)$

$$\begin{aligned}
f'(a) &= F_a(a) \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} dz\, \frac{F_a(z)}{z-a} \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} dz\, \frac{f(z)}{(z-a)^2} - f(a)\frac{1}{2\pi i}\oint_{\mathcal{C}} dz\, \frac{1}{(z-a)^2} \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} dz\, \frac{f(z)}{(z-a)^2}
\end{aligned}$$

This gives an expression for $f'(a)\ \forall a \in \mathcal{D}$.

Analyticity of $f'(z)$:

$$f'(a+\xi) - f(a) = \left[\frac{2}{2\pi i}\oint_{\mathcal{C}} dz\, \frac{f(z)}{(z-a-\xi)^2\,(z-a)}\right]\xi$$

where terms of order ξ^2 have been dropped. In the limit ξ is infinitesimal, we can ignore ξ in the integral as well. Then

$$f'(a+\xi) - f(a) = \left[\frac{2}{2\pi i}\oint_{\mathcal{C}} dz\, \frac{f(z)}{(z-a)^3}\right]\xi$$

Then $f'(z)$ scales and rotates ξ by a factor independent of ξ and is therefore analytic, with $f''(z)$ given by

$$f''(a) = \frac{2}{2\pi i} \oint_{\mathcal{C}} dz \, \frac{f(z)}{(z-a)^3}$$

We use mathematical induction. The result is clearly true for $n = 1$. Let $f^n(a)$ exist, given by

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{C} dz \, \frac{f(z)}{(z-a)^{n+1}}$$

Then

$$
\begin{aligned}
f^{(n)}(a+\xi) - f^{(n)}(a) &= \frac{n!}{2\pi i} \oint_C dz\, f(z) \frac{\left[(z-a)^{n+1} - (z-a-\xi)^{n+1}\right]}{(z-a-\xi)^{n+1}\,(z-a)^{n+1}} \\
&\approx \frac{n!}{2\pi i} \oint_C dz\, f(z) \left[\frac{(n+1)\,\xi}{(z-a-\xi)^{n+1}\,(z-a)}\right]
\end{aligned}
$$

where binomial theorem has been used and terms of order ξ^2 and higher dropped. Dropping ξ in the denominator gives

$$
f^{(n)}(a+\xi) - f^{(n)}(a) = \frac{(n+1)!}{2\pi i} \oint_C dz \left[\frac{f(z)}{(z-a)^{n+2}}\right] \xi
$$

which shows that $f^{(n)}(z)$ is analytic with derivative given by the integral formula.

Taylor Series

Theorem

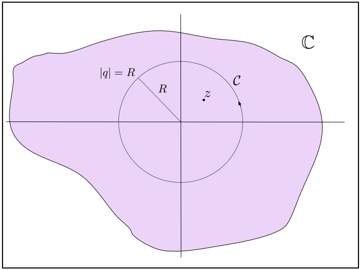
If a function $f(z)$ is analytic within an origin centered disc of radius R then at any point z within the disc it can be expressed as a series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots.$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}$$

Proof



$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dq \, \frac{f(q)}{q - z}$$

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} dq\, \frac{f(q)}{q-z} \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} dq\, \frac{f(q)}{q}\left[\frac{1}{1-(z/q)}\right] \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} dq\, \frac{f(q)}{q}\left[1+(z/q)+(z/q)^2+..\right] \\
&= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n
\end{aligned}$$

where the integral formula for $f^{(n)}(0)$ has been used.

Generalization:

Theorem

If a function $f(z)$ is analytic within a disc of radius R centered around a point z_0 then at any point z within the disc it can be expressed as a series

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \dots + c_n(z - z_0)^n + \dots.$$

where

$$c_n = \frac{f^{(n)}(z_0)}{n!}$$

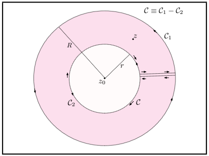
Laurent Series

Theorem

A function $f(z)$ which is analytica in an annular region $r < |z - z_0| < R$ centered about a point z_0 can always be expanded in a series ($\forall z$ in the annular region) as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Proof



$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{\mathcal{C}} dz' \, \frac{f(z')}{z'-z} \\ &= \frac{1}{2\pi i}\oint_{\mathcal{C}_1} dz' \, \frac{f(z')}{z'-z} - \frac{1}{2\pi i}\oint_{\mathcal{C}_2} dz' \, \frac{f(z')}{z'-z} \\ &= S_1 + S_2 \end{aligned}$$

$$
\begin{aligned}
S_1 &= \frac{1}{2\pi i} \oint_{\mathcal{C}_1} dz' \, \frac{f(z')}{z' - z} \\
&= \frac{1}{2\pi i} \oint_{\mathcal{C}_1} dz' \, \frac{f(z')}{(z' - z_0) - (z - z_0)} \\
&= \frac{1}{2\pi i} \oint_{\mathcal{C}_1} dz' \, \frac{f(z')}{(z' - z_0)\left[1 - (z - z_0) / (z' - z_0)\right]} \\
&= \frac{1}{2\pi i} \oint_{\mathcal{C}_1} dz' \, \frac{f(z')}{(z' - z_0)} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(z' - z_0)^n}
\end{aligned}
$$

since $|z - z_0| < |z' - z_0|$ on $\mathcal{C}_1$

Therefore

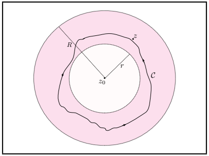
$$S_1 = \sum_{n=0}^{\infty} a_n \left(z - z_0\right)^n ; \quad a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}_1} dz' \, \frac{f(z')}{\left(z' - z_0\right)^{n+1}}$$

Similarly, on $\mathcal{C}_2, \ |z' - z_0| < |z - z_0|$. Therefore

$$\begin{aligned} S_2 &= \frac{1}{2\pi i} \oint_{\mathcal{C}_1} dz' \, \frac{f(z')}{(z - z_0)} \sum_{n=0}^{\infty} \frac{\left(z' - z_0\right)^n}{\left(z - z_0\right)^n} \\ &= \sum_{n=-1}^{-\infty} a_n \left(z - z_0\right)^n \end{aligned}$$

where again $a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}_1} dz' \, f(z') / \left(z' - z_0\right)^{n+1}$.

Finally



$$f(z) = \sum_{n=-\infty}^{\infty} a_n \left(z - z_0\right)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz' \, \frac{f(z')}{\left(z' - z_0\right)^{n+1}}$$

Example

$f(z) = 1/\left[z(z-1)\right]$ is analytic in the region $0 < |z| < 1$. It can therefore be expanded in a Laurent Series.

$$\begin{aligned} f(z) &= \frac{1}{z(z-1)} \\ &= -\frac{1}{1-z} - \frac{1}{z} \\ &= -\frac{1}{z} - \left(1 + z + z^2 + z^3 + \ldots\right) \\ &= -\frac{1}{z} - 1 - z - z^2 - z^3 - \ldots. \end{aligned}$$

Example

From the general expression

$$
\begin{aligned}
a_n &= \frac{1}{2\pi i}\oint_{\mathcal{C}} dz' \, \frac{f(z')}{(z'-z_0)^{n+1}}; \quad z_0 = 0 \\
&= \frac{1}{2\pi i}\oint_{\mathcal{C}} dz' \, \frac{[1/(z'-1)]}{(z')^{n+2}} \\
&= \begin{cases} -1, & n \geq -1 \\ 0, & n < -1 \end{cases}
\end{aligned}
$$

Problem

Expand $f(z) = 1/\left[z(z-1)\right]$ *in a Laurent Series about* $z = 1$.

Singularities

Definition

A function $f(z)$ is said to have an isolated singularity at $z = z_0$ if it is not analytic at z_0 but is analytic at all neighboring points.

A function can always be expanded in a Laurent Series about an isolated singular point.

Definition

If there exists a term $1/(z-z_0)^m$ in the Laurent expansion of $f(z)$ with most negative power of $z-z_0$ then $f(z)$ is said to have a 'pole of order m' at $z=z_0$. Else, it is said to have an 'Essential Singularity' at $z=z_0$.

Definition

The 'residue' of $f(z)$ at an isolated singular point z_0 is defined to be the coefficient of $1/(z-z_0)$ in the Laurent expansion of $f(z)$ about $z=z_0$.

Residue Theorem

Theorem

Let $f(z)$ be analytic on and within a closed contour $\mathcal{C}$ except at isolated singular points $z_1, z_2, ..z_k$. Then

$$\oint_{\mathcal{C}} dz\ f(z) = 2\pi i \sum_{i=1}^{k} \nu(z_i)\, Res\,[z_i]$$

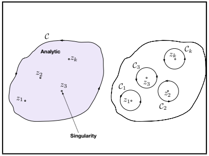
where $\nu(z_i)$ is the winding number of the contour for point z_i and $Res\,[z_i]$ is the residue of $f(z)$ at $z = z_i$.

Proof for a single singular point and a simple contour: $f(z)$ can be expanded in a Laurent Series about the pole z_i. Then,

$$\begin{aligned}\oint_{\mathcal{C}} dz\, f(z) &= \sum_{n=-\infty}^{\infty} a_n \oint_{\mathcal{C}} dz\,(z - z_0)^n \\ &= a_{-1} \oint_{\mathcal{C}} dz\,(z - z_0)^{-1} \\ &= 2\pi i \times a_{-1} \\ &= 2\pi i\, Res\,[z_i]\end{aligned}$$

where a_{-1} is the expansion coefficient for the term $1/\,(z - z_i)$ in the Laurent series for $f(z)$.

Proof for simple closed contour but multiple poles:



$$\oint_{\mathcal{C}} dz\ f(z) = \sum_{i=1}^{k} \oint_{\mathcal{C}_i} dz\ f(z)$$

$$= 2\pi i \sum_{i=1}^{k} Res\,[z_i]$$

Calculation of Residue

Pole of order m:

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \ldots + a_0 + a_1 (z - z_0) + \ldots$$

$$\therefore (z - z_0)^m f(z) = a_{-m} + a_{-m+1} (z - z_0) + .. + a_{-1} (z - z_0)^{m-1} + ..$$

This gives

$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$$

Example

Residue of $f(z) = 1/\sin z$ at $z = 0$:

$$\frac{1}{\sin z} = \frac{1}{z - z^3/3! + z^5/5! + \ldots}$$

$f(z)$ has a pole of order 1 at $z = 0$. Then

$$\begin{aligned} a_{-1} &= \lim_{z \to 0} \frac{z}{\sin z} \\ &= 1 \end{aligned}$$

Example

$f(z) = e^{-1/z}$. This has an essential singularity at $z = 0$. However, in terms of $w = 1/z$, it is analytic at $w = 0$ and so has the Taylor expansion

$$\begin{aligned} e^{-w} &= 1 - w + w^2/2! - w^3/3! + \ldots \\ &= 1 - 1/z + 1/\left(2!\, z^2\right) + \ldots \end{aligned}$$

Then, $a_{-1} = -1$.

Trigonometric Integrals

Integrals of the form

$$I = \int_0^{2\pi} d\theta\ f(\sin\theta, \cos\theta)$$

where f is finite $\forall\ \theta$ and is a rational function of $\sin\theta$ and $\cos\theta$. Using $z = e^{i\theta}$,

$$I = -i \oint_{\mathcal{C}} \frac{dz}{z} \left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2} \right)$$

where $\mathcal{C}$ is the unit circle. $I = (-i)2\pi i \sum$ residues of f within $\mathcal{C}$.

Example

$$
\begin{aligned}
I &= \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1 \\
&= -i\,\frac{2}{a} \oint_{\mathcal{C}} \frac{dz}{z^2 + (2/a)\,z + 1}
\end{aligned}
$$

The integand has poles at $z_1 = -\left(1 + \sqrt{1 - a^2}\right)/a$ and $z_2 = -\left(1 - \sqrt{1 - a^2}\right)/a$ of which z_2 is within $\mathcal{C}$. This gives $I = (-2i/a)\,2\pi i\,[1/(z_2 - z_1)] = 2\pi/\sqrt{1 - a^2}$.

Problem

Evaluate

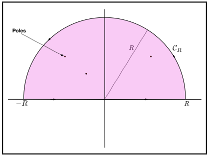
$$I = \int_0^{2\pi} \frac{d\theta \ \cos 2\theta}{5 - 4\cos\theta}$$

Integrals With Range $-\infty$ to ∞

$$I = \int_{-\infty}^{\infty} dx\, f(x)$$

Assumptions:

- $f(z)$ is analytic in upper/lower half of complex plane, including the real axis, except for a finite number of poles.
- As $|z| \to \infty,\ f(z) \to 0$ faster than $1/z$.



$$\oint_{\mathcal{C}} dz\, f(z) = \int_{-R}^{R} dx\, f(x) + \int_{\mathcal{C}_R} dz\, f(z)$$

$$\lim_{R\to\infty} \int_{\mathcal{C}_R} dz\, f(z) = 0$$

Then

$$\begin{aligned} \int_{-\infty}^{\infty} dx\, f(x) &= \lim_{R\to\infty} \oint_{\mathcal{C}} dz\, f(z) \\ &= 2\pi i \sum Res\, f(z) \end{aligned}$$

Example

$$\begin{aligned} I &= \int_0^\infty \frac{dx}{1+x^2} \\ &= \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^2} \end{aligned}$$

$f(z) = 1/\left(1+z^2\right) = 1/\left[(z+i)(z-i)\right]$. Then $Res_{z=i} f(z) = 1/(2i)$. Therefore

$$\begin{aligned} I &= 2\pi i \times \frac{1}{2i} \\ &= \frac{\pi}{2} \end{aligned}$$

Problem

Evaluate

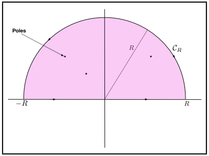
$$I = \int_0^\infty \frac{dx}{1 + x^4}$$

Complex Exponentials

$$I = \int_{-\infty}^{\infty} dx\ e^{iax}\ f(x); \quad a > 0$$

Assumptions:

- $f(z)$ is analytic in upper half of complex plane, including the real axis, except for a finite number of poles.
- As $|z| \to \infty, \ f(z) \to 0$ in upper half plane.



$$\oint_{\mathcal{C}} dz\, f(z)\, e^{iaz} = \int_{-R}^{R} dx\, f(x)\, e^{iax} + \int_{\mathcal{C}_R} dz\, f(z)\, e^{iaz}$$

$$\lim_{R\to\infty} \int_{\mathcal{C}_R} dz\, f(z)\, e^{iaz} = 0 \ \text{(Jordan's Lemma)}$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} dx\, f(x)\, e^{iax} &= \lim_{R\to\infty} \oint_{\mathcal{C}} dz\, f(z)\, e^{iaz} \\ &= 2\pi i \sum Res\, f(z)\, e^{iaz}\end{aligned}$$

Jordan's Lemma

$$I_R = \int_{\mathcal{C}_R} dz\, f(z)\, e^{iaz} = iR \int_0^{\pi} d\theta\, e^{i\theta}\, f\left(Re^{i\theta}\right)\, e^{iaR\cos\theta - aR\sin\theta}$$

Then

$$\begin{aligned} |I_R| &\leq R\, |f|_{\max} \int_0^{\pi} d\theta\, e^{-aR\sin\theta} \\ &= 2R\, |f|_{\max} \int_0^{\pi/2} d\theta\, e^{-aR\sin\theta} \end{aligned}$$

For $\theta \in [0, \pi/2]$, $2\theta/\pi \leq \sin\theta$. Then

$$\begin{aligned} |I_R| &\leq 2R\, |f|_{\max} \int_0^{\pi/2} d\theta\, e^{-2aR\theta/\pi} \\ &= 2R\, |f|_{\max} \frac{1 - e^{-aR}}{2aR/\pi} < \frac{\pi}{a} |f|_{\max} \to 0 \text{ as } R \to \infty \end{aligned}$$

Example

$$I = \int_0^\infty \frac{dx \ \cos x}{1 + x^2}$$

Using $\cos x = \left(e^{ix} + e^{-ix}\right)/2$, we get

$$I = \frac{1}{2} \int_{-\infty}^\infty \frac{dx \ e^{ix}}{1 + x^2}$$

Then, $f(z) = 1/2\left(z^2 + 1\right)$ with pole at $z = i$ in upper half plane. Simple calculation gives $I = \pi/2e$.