Mobius Transformations

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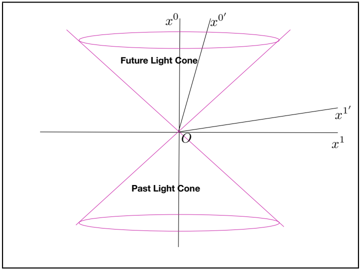
Outline

1. Special Relativity
2. Lorentz Transformations and SL(2,C)
3. The Celestial Sphere
4. Mobius Transformations

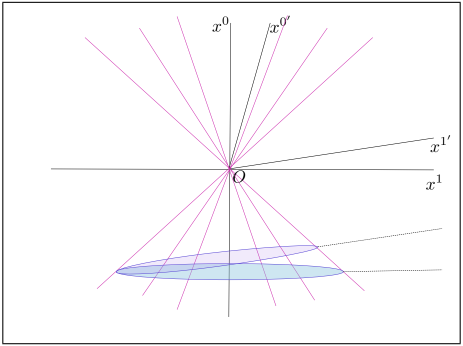
Spacetime

A point in spacetime is assigned four coordinates (x^0, x^1, x^2, x^3) by inertial observers. The coordinates are related by Lorentz Transformations such that

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x^{0'})^2 - (x^{1'})^2 - (x^{2'})^2 - (x^{3'})^2$$



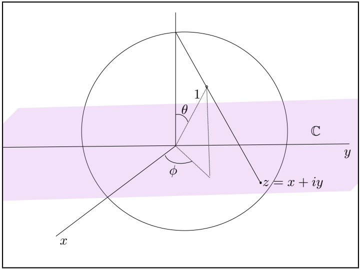
The ‘Celestial Sphere’



Different observers define different Celestial Spheres (Relativity of Simultaneity).

Mapping the Celestial Sphere to $\mathbb{C}$

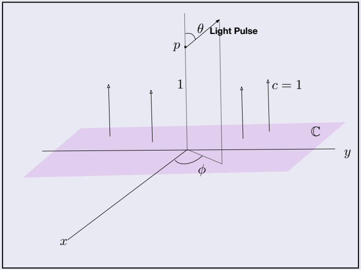
Celestial Sphere formed at unit time by a pulse of light emitted from the origin at $x^0 = x^{0'} = 0$



Problem

Show that $z = \cot(\theta/2)\, e^{i\phi}$.

Problem



The origin of the complex plane is one unit below a point which emits a pulse of light. At the instant the pulse is emitted, the plane $\mathbb{C}$ starts moving up with speed $c = 1$. Referring to the illustration, show that a light ray emitted at (θ, ϕ) intersects $\mathbb{C}$ at $z = \cot(\theta/2)\, e^{i\phi}$.

Lorentz Transformations and $SL(2, C)$

Any spacetime point (x^0, x^1, x^2, x^3) can be represented by a 2×2 Hermitian matrix

$$X = \begin{pmatrix} x^0 - x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 + x^3 \end{pmatrix}$$

such that $X^\dagger = (X^T)^* = X$ and

$$\det X = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

Action of a Lorentz Transformation:

$$X' = Q\, X\, Q^\dagger$$

where Q is a 2×2 complex matrix.

Invariance of invariant interval: $\det X' = \det X$.

$$\begin{aligned} \det X' &= |\det Q|^2 \; \det X \\ \Longrightarrow \; |\det Q| &= 1 \end{aligned}$$

Choice $\det Q = 1$ gives a set of complex 2×2 matrices with unit determinant (the set $SL(2, C)$). The product of two elements of $SL(2, C)$ belongs to $SL(2, C)$. This set forms a 'group', representing the fact that Lorentz transformations in succession $\leftrightarrow$ single Lorentz transformation.

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 0$$

In spherical polar coordinates

$$X = \begin{pmatrix} x^0 - r\cos\theta & r\sin\theta e^{i\phi} \\ r\sin\theta e^{-i\phi} & x^0 + r\cos\theta \end{pmatrix}$$

Spacetimes points along light rays reaching the origin at $t = 0$ are given by $x^0 = -r$. For such points

$$\begin{aligned} X &= \begin{pmatrix} -r\,(1+\cos\theta) & r\sin\theta e^{i\phi} \\ r\sin\theta e^{-i\phi} & -r\,(1-\cos\theta) \end{pmatrix} \\ &= \begin{pmatrix} -2r\cos^2(\theta/2) & 2r\sin(\theta/2)\cos(\theta/2)\,e^{i\phi} \\ 2r\sin(\theta/2)\cos(\theta/2)\,e^{-i\phi} & -2r\sin^2(\theta/2) \end{pmatrix} \end{aligned}$$

Image Recorded by a Camera

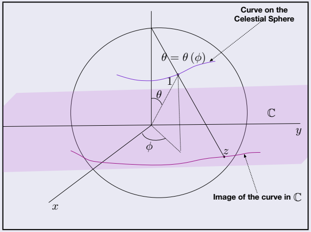
A camera image does not distinguish between events along a particular light ray. Along a direction (θ, ϕ), choose $r = 1/\left(2\sin^2(\theta/2)\right)$ (the point from which we pretend light was emitted which reaches the origin at $t = 0$). The matrix corresponding to the camera recording is then

$$\begin{aligned} X &= \begin{pmatrix} -\cot^2(\theta/2) & \cot(\theta/2)\, e^{i\phi} \\ \cot(\theta/2)\, e^{-i\phi} & -1 \end{pmatrix} \\ &= \begin{pmatrix} -|z|^2 & z \\ \bar{z} & -1 \end{pmatrix} \end{aligned}$$

where $z = \cot(\theta/2)\, e^{i\phi}$.

Consequence

The image of the celestial sphere recorded by a camera can be mapped by a Stereographic Projection to $\mathbb{C}$



Problem

Show that stereographic projection is conformal, i.e., preserves angles between curves locally.

Effect of Lorentz Transformation

$$\begin{aligned} X' &= Q\,X\,Q^\dagger \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -\left|z\right|^2 & z \\ \bar{z} & -1 \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{aligned}$$

where $\alpha = -\left|z\right|^2 \left|a\right|^2 + b^* az + ba^* \bar{z} - \left|b\right|^2$,
$\beta = -\left|z\right|^2 ac^* + zd^* a + bc^* \bar{z} - bd^*$,
$\gamma = -\left|z\right|^2 a^* c + \bar{z} da^* + b^* cz - b^* d$ and
$\delta = -\left|z\right|^2 \left|c\right|^2 + d^* cz + dc^* \bar{z} - \left|d\right|^2$

Dividing throughout by $-\delta$, we get

$$X' \to \begin{pmatrix} -\alpha/\delta & -\beta/\delta \\ -\gamma/\delta & -1 \end{pmatrix}$$

Problem

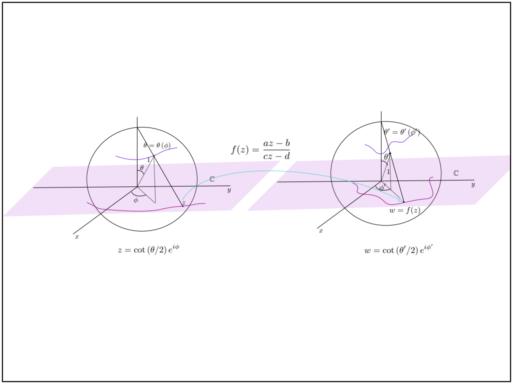
Verify that

$$X' = \begin{pmatrix} -|w|^2 & w \\ \bar{w} & -1 \end{pmatrix}$$

where

$$w = \frac{az-b}{cz-d}$$

The celestial sphere of observer O' is related to w by $w = \cot(\theta'/2)\, e^{i\phi'}$. Given a curve $\theta = \theta(\phi)$, the map $w = (az - b)\,/\,(cz - d)$ allows us to determine the equation $\theta' = \theta'(\phi')$.



Mobius Transformations

$$f(z) = \frac{az+b}{cz+d}$$

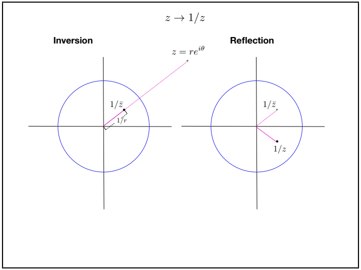
Sequence of transformations:

- $z \to z + \frac{d}{c}$: Translation
- $z \to (1/z)$: Reciprocation
- $z \to -\frac{(ad-bc)}{c^2}\, z$: Scaling + Rotation
- $z \to z + \frac{a}{c}$: Translation

All except Reciprocation preserve shapes.

Inversion in Unit Circle

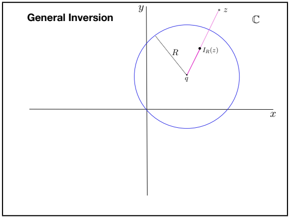
Reciprocation = Inversion + Reflection
Inversion: $I(z) = 1/\bar{z}$



Inversion swaps points within and without the unit circle. The unit circle is invariant.

General Inversion

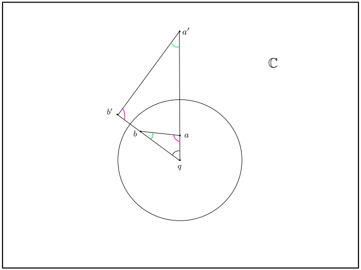
Generalization to circle of radius R centered around $z = q$: A point at distance ρ from center of circle should be mapped to a point at distance R^2/ρ



Problem

Show that

$$I_R(z) = \frac{R^2}{(\bar{z} - \bar{q})} + q$$



$$|a' - q| = \frac{R^2}{|a - q|}$$

$$|b' - q| = \frac{R^2}{|b - q|}$$

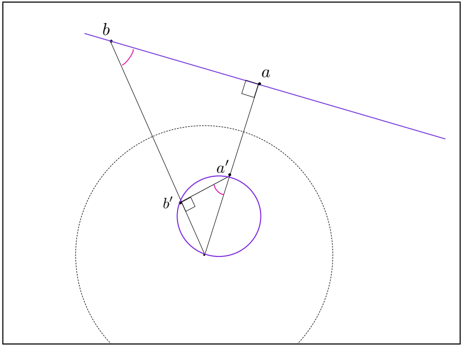
$$\Longrightarrow \frac{|a - q|}{|b - q|} = \frac{|b' - q|}{|a' - q|} \Longrightarrow \Delta aqb \sim \Delta b'qa'$$

Problem

Show that $|a' - b'| = \left(\frac{R^2}{|q-a|\ |q-b|}\right)|a - b|$

Property

If a line does not pass through the center of a circle $\mathcal{K}$, its inversion in $\mathcal{K}$ is a circle passing through the center of $\mathcal{K}$.



Property

If the image of a line under inversion in a circle $\mathcal{K}$ is a circle passing through the center, the same is true for any other circle.

Proof.

Let the image of a point z in circle $\mathcal{K}_1$ of radius R_1 be z_1 and in circle $\mathcal{K}_2$ of radius R_2 be z_2. Then

$$\frac{|z_2 - q|}{|z_1 - q|} = \frac{R_2^2}{R_1^2}$$

then

$$I_{K_2} = D_{\left(R_2^2/R_1^2\right)} \cdot I_{K_1}$$

where I_{K_2}, I_{K_1} are inversions and D is dilation (scaling). Dilation preserves the property, so it is true for all circles.

Property

Image of a circle under inversion is a circle.