Analytic Functions

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Outline

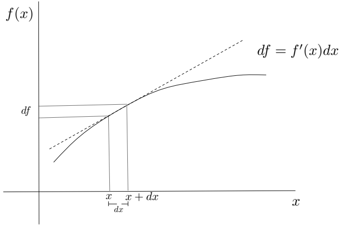
The Complex Derivative

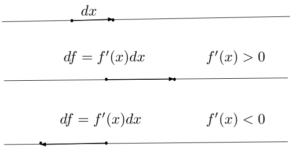
Mystery

Complex functions discussed so far map ‘small squares to small squares’. What is the significance of this?

Such maps are called ’Conformal’ maps. They preserve angles locally. Is there a general way to construct other such maps using complex functions?

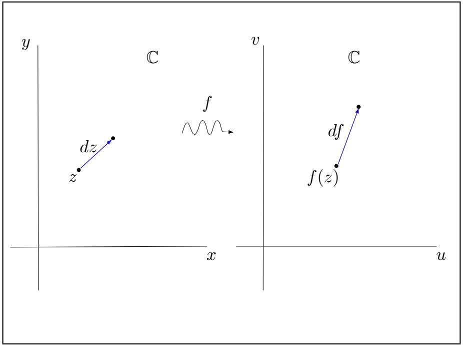
Visual differentiation of a real function



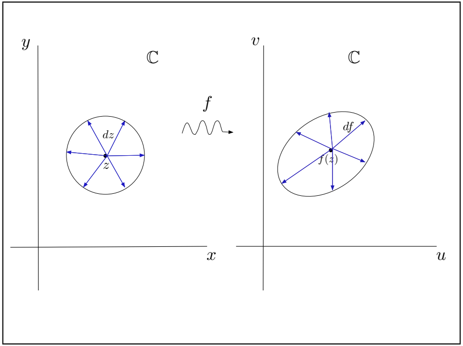


Generalization to $\mathbb{C}$:

$$df = f'(z)dz$$

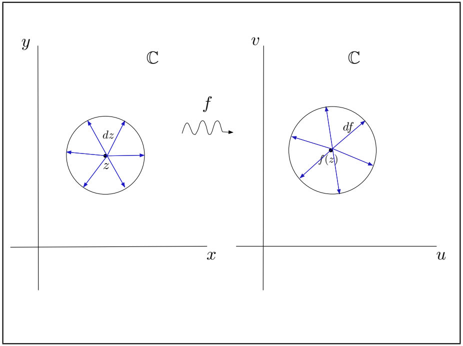


General complex map distorts a small region:

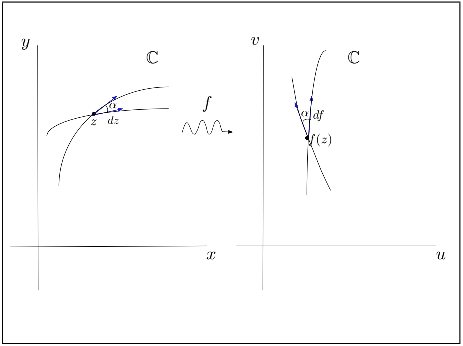


$f'(z)$ is not defined for such a map.

Analytic map: $f'(z)$ exists. $f'(z)$ will produce a local scaling plus rotation, same for all dz



Analytic map preserves angles locally, since every dz located at z is rotated by $\arg f'(z)$. However, the region can be locally stretched/contracted



Example

$f(z) = z + a$ is trivially analytic. Since $df = dz$, $f'(z) = 1$.

Example

$f(z) = a \cdot z$. This is analytic, since dz located at z will be scaled by $|a|$ and rotated by $\arg a$. Since $df = a \cdot dz$, $f'(z) = a$.

Example

$f(z) = z^n$.
$z = re^{i\theta} \implies f(z) = r^n e^{in\theta}$. Therefore,
$u = r^n \cos n\theta,\ v = r^n \sin n\theta$. After some work,

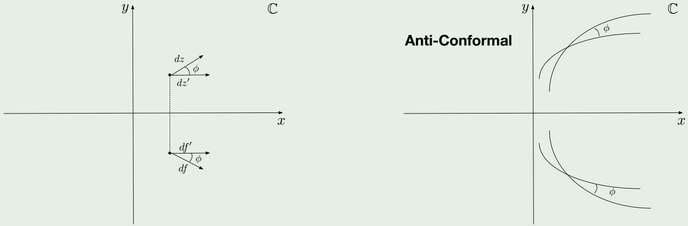
$$\begin{aligned} du &= nr^{n-1}\sin(n-1)\theta\ dx - nr^{n-1}\sin(n-1)\theta\ dy \\ dv &= nr^{n-1}\cos(n-1)\theta\ dx + nr^{n-1}\cos(n-1)\theta\ dy \end{aligned}$$

This is a local scaling by nr^{n-1} and a rotation by $(n-1)\theta$. Therefore, it preserves angles locally and is analytic.

$$\begin{aligned} f'(z) &= n\,r^{n-1}e^{(n-1)\theta} \\ &= n\,z^{n-1} \end{aligned}$$

Example

$f(z) = \bar{z}$. Is this analytic?
Note: $\bar{z}$ is a reflection of z in the real axis.



Angle of rotation depends on the orientation. So, $f(z) = \bar{z}$ is not analytic.

Cauchy-Riemann Equations

$f(z) = u(x, y) + i\ v(x, y)$. What are the constraints on functions u and v that ensure analyticity/conformality?

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

To locally be a scaling plus rotation, the Jacobian matrix must have the form

$$J = \lambda \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Conditions for analyticity:

Caucy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Given

$$df = du + idv$$
$$= f'(z)\,dz$$

this allows us to deduce $f'(z)$

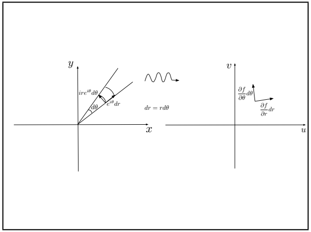
Complex Derivative

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial f}{\partial x} \end{aligned}$$

Alternative Form

$$\begin{aligned} f'(z) &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \\ &= -i\frac{\partial f}{\partial y} \end{aligned}$$

Polar form of Cauchy-Riemann Equations



$$\frac{\partial f}{\partial \theta}\, d\theta = i\frac{\partial f}{\partial r} r\, d\theta$$

$$\implies \frac{\partial f}{\partial \theta} = i\, r\, \frac{\partial f}{\partial r}$$

Polar Form

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$
$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\begin{aligned} df &= f'(z)dz \\ \implies f'(z) &= e^{-i\theta}\frac{\partial f}{\partial r} \\ &= \frac{-i}{z}\frac{\partial f}{\partial \theta} \end{aligned}$$

Problem

Show that $f(z) = z^n$ is analytic and find $f'(z)$ using the polar form.

Problem

Writing $f(z) = R\, e^{i\psi}$, show that the Cauchy-Riemann equations are equivalent to

$$\begin{aligned} \frac{\partial R}{\partial \theta} &= -r\, R\, \frac{\partial \psi}{\partial r} \\ R\, \frac{\partial \psi}{\partial \theta} &= r\, \frac{\partial R}{\partial r} \end{aligned}$$

Differentiation Rules

Given $f(z)$ and $g(z)$ are analytic over some domain, their sum is analytic over that domain.

$$\begin{aligned} h(z) &= f(z) + g(z) \\ \implies h'(z) &= f'(z) + g'(z) \end{aligned}$$

Proof:

$$\begin{aligned} dh &= h(z+dz) - h(z) \\ &= [f(z+dz) - f(z)] + [g(z+dz) - g(z)] \\ &= df + dg \\ &= \left[f'(z) + g'(z)\right] dz \end{aligned}$$

Given $f(z)$ and $g(z)$ are analytic over some domain, their product is analytic over that domain.

$$\begin{aligned} h(z) &= f(z)\, g(z) \\ \Longrightarrow h'(z) &= f(z)\, g'(z) + g(z)\, f'(z) \end{aligned}$$

Proof:

$$\begin{aligned} dh &= h(z+dz) - h(z) \\ &= f(z+dz)\, g(z+dz) - f(z)\, g(z) \\ &= \left[f(z) + f'(z)\, dz\right] \left[g(z) + g'(z)\, dz\right] - f(z)\, g(z) \\ &= \left[f(z)\, g'(z) + g(z)\, f'(z)\right] dz \end{aligned}$$

where terms of order dz^2 are dropped.

Quotient Rule

If $f(z)$ and $g(z)$ are analytic on some domain, their ratio is analytic everywhere on the domain except at singular points.

Problem

Show that if $h(z) = f(z)/g(z)$ then

$$h'(z) = \frac{g(z)\, f'(z) - f(z)\, g'(z)}{g^2(z)}$$

Composition Rule

If $f(z)$ is analytic over some domain and $g(z)$ is analytic over its image then $g(f(z))$ is analytic over the domain of f.

Composition

$$g'(f(z)) = g'(w)f'(z)$$

where $w = f(z)$.

Proof: Let $h(z) = g(f(z))$. Then,

$$\begin{aligned} h(z+dz) &= g(f(z+dz)) \\ &= g\left(f(z) + f'(z)dz\right) \quad \text{(since } f'(z) \text{ exists)} \\ &= g(f(z)) + g'(w)f'(z)dz \quad \text{(since } g'(w) \text{ exists)} \\ \implies dh &= \left(g'(w)f'(z)\right) dz \end{aligned}$$

Consequences of Differentiation Rules:

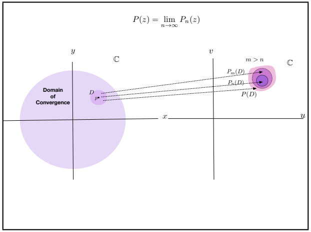
- $z^n = z \cdot z \cdot z....z \cdot z$ is analytic and $(z^n)' = n\, z^{n-1}$
- Polynomials $P_n(z) = c_0 + c_1 z + c_2 z^2 + .. + c_n z^n$ are analytic and

$$P_n'(z) = c_1 + 2\ c_2\ z + 3\ c_3\ z^2 + .. + n\ c_n\ z^{n-1}$$

- Power Series - are they analytic ??

Analyticity of Power Series

Let $P(z) = c_0 + c_1\ z + c_2\ z^2 + c_3\ z^3 + \ldots$ converge over some domain. $P(z)$ is the limit of the sequence $P_n(z) = c_0 + c_1\ z + c_2\ z^2 + \ldots + c_n\ z^n$.



$P(z)$ maps small a small disc to a small disc centered about any point z within radius of convergence.

Analyticity of Power Series

A Power Series $P(z)$ is analytic at all points within its radius of convergence.

Therefore, $P'(z)$ exists and is a limit of $P'_n(z)$. Since $P'_n(z)$ is also a polynomial, it is analytic and preserves small discs. Therefore, $P'(z)$ preserves small discs and is therefore also analytic.

Consequence

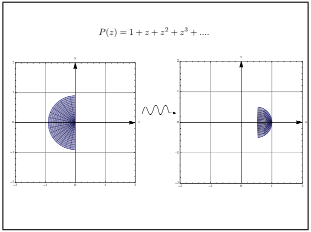
A Power Series is infinitely differentiable within its radius of convergence.

Analytic Continuation

Consider the power series

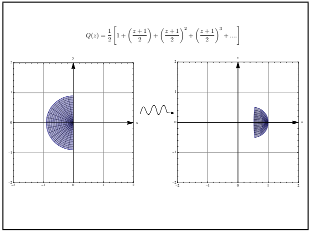
$$P(z) = 1 + z + z^2 + z^3 + ...$$

with domain of convergence $|z| < 1$. In this region, it defines an analytic function $P(z)$, which preserves angles. However, this function is defined only for $|z| < 1$.



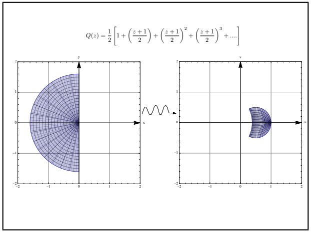
It is clear we can geometrically extend the image such that it is conformal. Consider a different power series and the image of the same region in $\mathbb{C}$

$$Q(z) = \frac{1}{2}\left[1 + \left(\frac{z+1}{2}\right) + \left(\frac{z+1}{2}\right)^2 + \left(\frac{z+1}{2}\right)^3 + ..\right], \quad |z+1| < 2$$



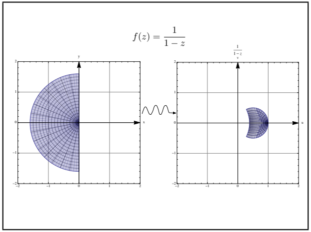
The image is the same.

The series $Q(z)$ converges in a larger region and ‘extends’ the action of $P(z)$ to new regions of $\mathbb{C}$. We say that $Q(z)$ is the ‘Analytic Continuation of $P(z)$ into the new region



Clearly, we can increase the image still further maintaining conformality. Then, there must exist further continuation of $P(z)$ and $Q(z)$ to other egions of $\mathbb{C}$.

Function $f(z) = \frac{1}{1-z}$ has the same action as $Q(z)$ and $P(z)$ over their respective domains



Unlike $P(z)$ and $Q(z)$, $f(z)$ is defined over all of $\mathbb{C}$. Then, we say $f(z)$ is the analytic continuation of $P(z)$ to $\mathbb{C}$.

Question

Is this 'continuation' unique?

Theorem

The Analytic Continuation of a function $f(z)$ is unique

Core idea: If two analytic functions defined over some domain $\mathcal{D}$ are equal on even a segment of a curve lying in $\mathcal{D}$ then they are equal over entire $\mathcal{D}$.