Complex Functions

A. Gupta

[1] Department of Physics
St. Stephen's College

Outline

Convergence of Powers Series

Power series in z seem to converge to same functions of z as of their real counterparts

$$1 + z + z^2 + z^3 + z^4 .... = \frac{1}{1-z} \quad \forall |z| < 1$$

What are the complex analogs of other series?

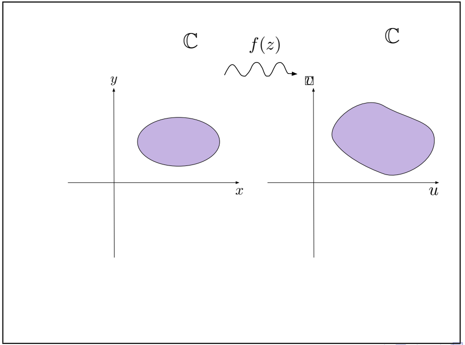
Example

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + ...$$

How do we interpret and ‘visualize’ e^z?

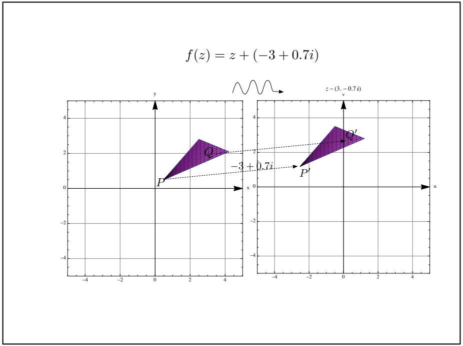
Complex Functions as Maps

Real functions can be visualized as 2-D graphical plots. For complex functions, we will need 4-D intuition! Simpler alternative: visualize $f(z)$ as a map $\mathbb{C} \rightarrow \mathbb{C}$.



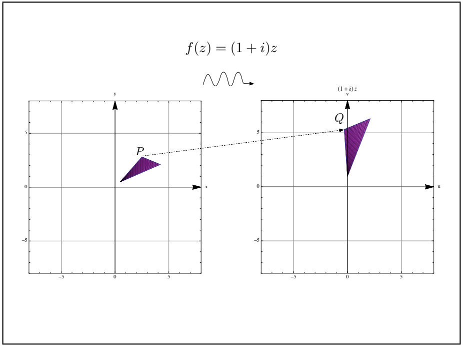
Translation

$$f(z) = z + a, \qquad a \in \mathbb{C}$$



Multiplication with a complex number

$$f(z) = a \cdot z, \qquad a \in \mathbb{C}$$



$$a = |a|\, e^{i\phi}, z = re^{i\theta}$$

$$a \cdot z = (|a|\, r)\, e^{i(\theta+\phi)}$$

Scaling by $|a| +$ rotation by ϕ about origin.
Figures retain shape but expand (or contract) and rotate.

Fun with Translations and Rotations

Translation operator T_a:

$$T_a(z) = z + a$$

Properties:

$$\begin{aligned} T_a^{-1} &= T_{-a} \\ T_a \cdot T_b &= T_{a+b} \end{aligned}$$

Rotation operator R_a^θ:
Rotates z about point a by θ

$$R_0^\theta(z) = e^{i\theta} z : \quad \text{Rotation about origin by } \theta$$

$$R_a^\theta = T_a \, R_0^\theta \, T_a^{-1}$$

$$\begin{aligned} R_a^\theta(z) &= e^{i\theta}(z - a) + a \\ &= e^{i\theta} z + k, \quad k = a\left(1 - e^{i\theta}\right) \end{aligned}$$

Rotation about a point $\leftrightarrow$ Rotation about origin + Translation

Problem

Show that rotation about the origin followed by a translation is equivalent to rotation about a point.

Problem

Show that translation followed by rotation about the origin is equivalent to rotation about a point.

Successive rotations

$$
\begin{aligned}
R_b^\phi \cdot R_a^\theta &= R_b^\phi \left[e^{i\theta} z + a \left(1 - e^{i\theta} \right) \right] \\
&= e^{i\phi} \left[e^{i\theta} z + a \left(1 - e^{i\theta} \right) \right] + b \left(1 - e^{i\phi} \right) \\
&= e^{i(\theta+\phi)} z + v, \quad v = a\, e^{i\phi} \left(1 - e^{i\theta} \right) + b \left(1 - e^{i\phi} \right)
\end{aligned}
$$

If $\theta + \phi$ is not a multiple of 2π, successive rotations $\leftrightarrow$ single rotation R_c^α by angle α about a point c.

Problem

If $\theta + \phi$ is an integral multiple of 2π, successive rotations are equivalent to a translation.

Problem

Let $M = R_{a_n}^{\theta_n} \cdot R_{a_{n-1}}^{\theta_{n-1}} \cdot ... R_{a_2}^{\theta_2} \cdot R_{a_1}^{\theta_1}$ be a composition of n rotations and $\theta = \theta_1 + \theta_2 + ... \theta_n$ be the total angle of rotation. Then $M = R_c^{\theta}$ for some c. However, if $\theta = 2n\pi$ for integer n then $M = T_v$ for some v.

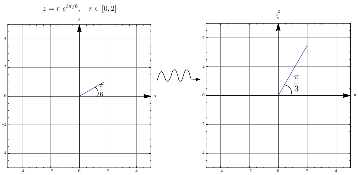
Integer Powers

Visualising $f(z) = z^n$

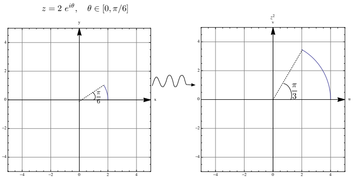
$$\begin{aligned} z &= r\, e^{i\theta} \\ f(z) &= r^n e^{in\theta} \end{aligned}$$

Action on rays and Arcs

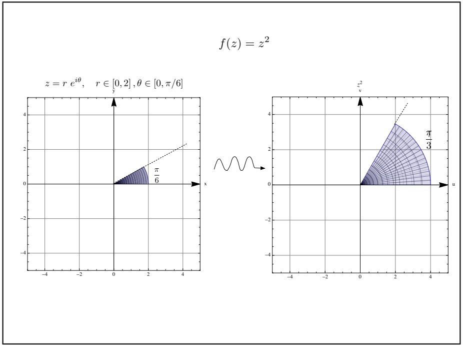
$f(z) = z^2$



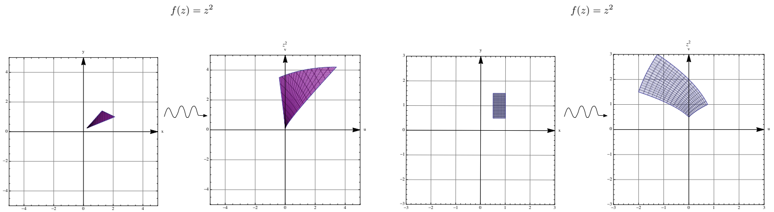
$f(z) = z^2$



Action on Sectors



Action on Triangles and Rectangles

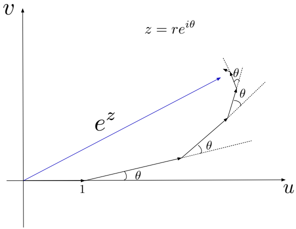


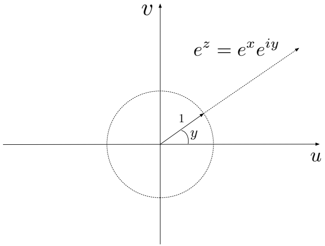
Observation: Triangles and Rectangles are not preserved by this transformation. However, 'small' rectanglular elements are preserved.

Exponential Function

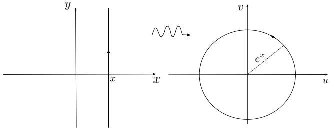
$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \ldots.$$

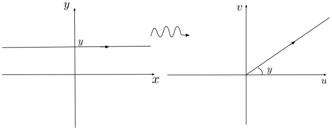
The series converges everywhere on $\mathbb{C}$, since it converges absolutely.

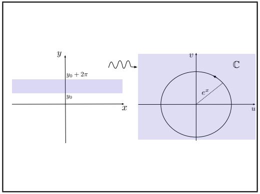




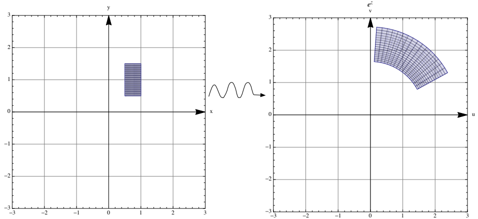
Geometry of mapping

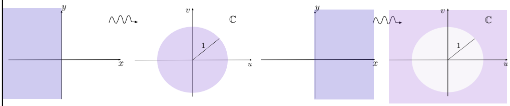






$$f(z) = e^z$$

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Sin/Cos/Hyperbolic Functions

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots.$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots.$$

$$\implies e^{iz} = \cos z + i \sin z \quad \text{Generalised Euler's formula}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Identities:

$$\begin{aligned}\cos^2 z + \sin^2 z &= (\cos z + i \sin z)(\cos z - i \sin z) \\ &= e^{iz} e^{-iz} \\ &= 1\end{aligned}$$

Problem

Show that

$$\begin{aligned}\cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b\end{aligned}$$

Mysterious Hyperbolic Functions

Real hperbolic functions:

$$\cosh x = \frac{e^x + e^x}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

Properties:

$$\begin{aligned} \cosh(a+b) &= \cosh a \cosh b + \sinh a \sinh b \\ \sinh(a+b) &= \sinh a \cosh b + \cosh a \sinh b \end{aligned}$$

Mystery: Why do hyperbolic functios share properties with harmonic functions?
Let us generalize to $\mathbb{C}$.

Complex hyperbolic functions:

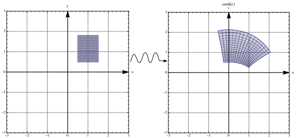
$$\cosh z = \frac{e^z + e^z}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

Observation:

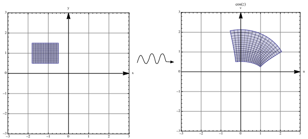
$$\cosh z = \cos(iz), \quad \sinh z = -i \sin(iz)$$

There is no real distinction between harmonic and hyperbolic functions in $\mathbb{C}$! cosh is just rotation of $\mathbb{C}$ by $\frac{\pi}{2}$ followed by cos. sinh is rotation of $\mathbb{C}$ by $\frac{\pi}{2}$, followed by sin and finally a rotation by $-\frac{\pi}{2}$.

$f(z) = \cosh z$



$f(z) = \cos z$



Problem

Show that:

$$\begin{aligned}
\cosh 2z &= \cosh^2 z + \sinh^2 z \\
\sinh 2z &= 2 \sinh z \cosh z \\
\cosh^2 z - \sinh^2 z &= 1
\end{aligned}$$