Complex power Series

A. Gupta

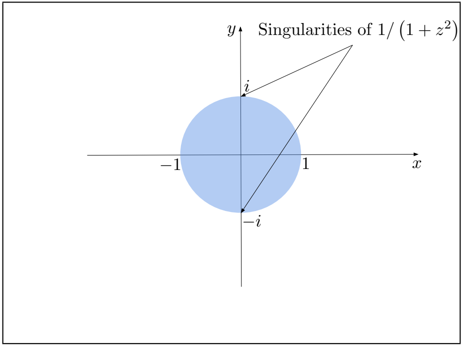
[1] Department of Physics
St. Stephen's College

Outline

Mysterious Divergence of Real Powers Series

Why is the following series representation defined only for $|x| < 1$?

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \ldots.$$



Complex Power Series

How do we interpret

$$f(z) = 1 - z^2 + z^4 - z^6 + z^8 \dots$$

Does this even ‘converge’?

Real Power Series

Limit of a Sequence

$$P_n(x) = c_0 + c_1 x + c_2 x^2 + .. + c_n x^n$$

$P_n(x)$ converges to $P(a)$ at $x = a$ if

$$\lim_{n \longrightarrow \infty} |P(a) - P_n(a)| = 0$$

Given $\epsilon > 0, \exists N$

$$|P(a) - P_n(a)| < \epsilon \quad \forall n > N$$

Cauchy Sequence

Given $\epsilon > 0, \exists N$

$$|P_n(a) - P_m(a)| < \epsilon \quad \forall n, m > N$$

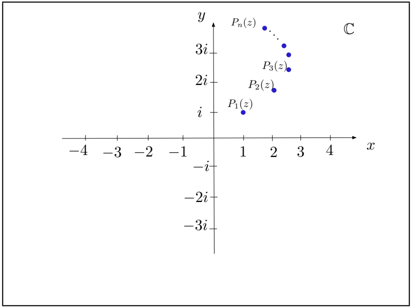
- Every convergent sequence in $\mathbb{R}$ is a Cauchy Sequence
- Every Cauchy Sequence in $\mathbb{R}$ converges in $\mathbb{R}$ ('Completeness' of $\mathbb{R}$)

Power Series in z

Limit of a Sequence

$$P_n(z) = c_0 + c_1 z + c_2 z^2 + .. + c_n z^n \qquad c_i \in \mathbb{C}$$

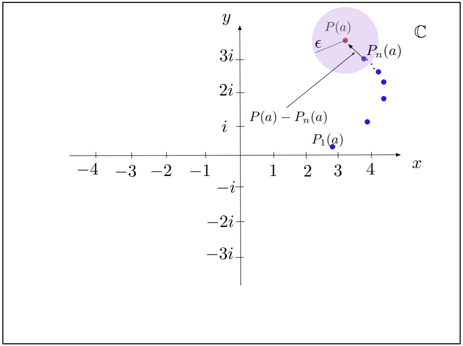
$P_n(z)$ is just a sequence of points in $\mathbb{C}$



Convergence in $\mathbb{C}$

Sequence $P_n(z)$ converges to a complex number $P(a)$ at $z = a$ if Given $\epsilon > 0, \exists N$

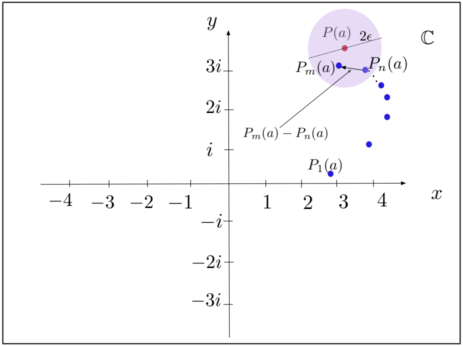
$$|P(a) - P_n(a)| < \epsilon \quad \forall n > N$$



Cauchy Sequence in $\mathbb{C}$

Given $\epsilon > 0, \exists N$

$$|P_n(a) - P_m(a)| < \epsilon \quad \forall n, m > N$$

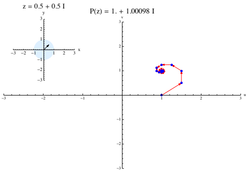


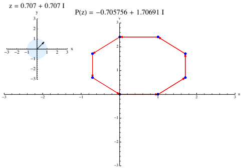
Exercise

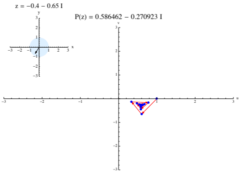
- Every convergent sequence in $\mathbb{C}$ is a Cauchy Sequence
- Every Cauchy Sequence in $\mathbb{C}$ converges in $\mathbb{C}$. Therefore, $\mathbb{C}$ is ‘complete’

Example

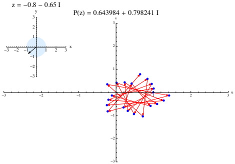
Sequence $P_n(z) = 1 + z + z^2 + z^3 + z^4 + ... + z^n$ We check for convergence at various points

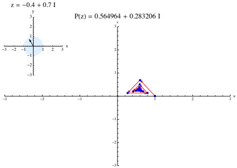


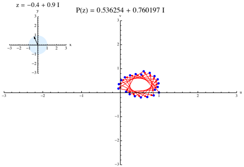




More points...







Observation: The series seems to converge at points within the circle $|z| < 1$ and diverge outside the circle.

Absolute Convergence

The comples power series $P(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + ...$ is said to converge 'absolutely' if the *real* series

$$\tilde{P}(z) = |c_0| + |c_1 z| + \left|c_2 z^2\right| + \left|c_3 z^3\right| + ..$$

converges.

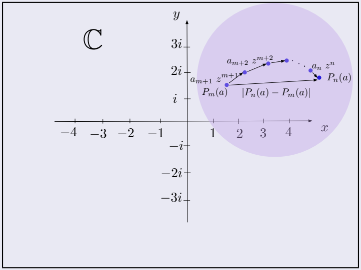
Theorem

If $\tilde{P}(a)$ converges, so does $P(a)$

Proof.

Outline of the proof: $\tilde{P}(a)$ forms a Cauchy Sequence. Therefore,

$$\left|\tilde{P}_n(a) - \tilde{P}_m(a)\right| < \epsilon \; \forall n, m > N$$



$$|P_n(a) - P_m(a)| < \left|\tilde{P}_n(a) - \tilde{P}_m(a)\right| < \epsilon \; \forall n, m > N$$

Theorem

If $P(z)$ converges at $z = a$, it absolutely converges everywhere in $|z| < |a|$

Proof.

Since P(a) converges, $|c_n a^n| < M \ \forall n$ for some M. Let $\rho = \frac{|z|}{|a|}$

$$
\begin{aligned}
\tilde{P}_n(z) - \tilde{P}_m(z) &= \left|c_{m+1} z^{m+1}\right| + \left|c_{m+2} z^{m+2}\right| + \ldots + |c_n z^n| \\
&= \rho^{m+1} \left|c_{m+1} a^{m+1}\right| + \ldots + \rho^n |c_n a^n| \\
&\leq M\left(\rho^{m+1} + \rho^{m+2} + \ldots \rho^n\right) \\
&= \frac{M}{1-\rho}\left(\rho^{m+1} - \rho^{n+1}\right)
\end{aligned}
$$

which can be made arbitrarily small for $\rho < 1$ for large n, m □

Corollary

If $P(z)$ diverges at some $z = a$, it will diverge for $|z| > |a|$

Theorem

Given a power series $P(z) = c_0 + c_1 z + c_2 z^2 + c_3 z^3 + ..$ there exists an $R > 0$ such that the series converges everywhere for $|z| < R$ and diverges for $|z| > R$

Theorem

Series expansion about $z = z_0$: Given the series $P(z) = c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + c_3 (z - z_0)^3 + ..$ there exists an $R > 0$ such that the series converges everywhere for $|z - z_0| < R$ and diverges for $|z - z_0| > R$