Multifunctions

A. Gupta

[1] Department of Physics
St. Stephen's College

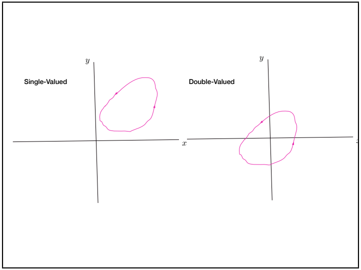
Outline

Complex Square Root

$$f(z) = z^{1/2}$$

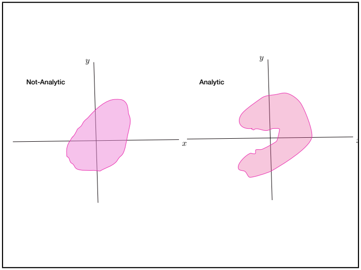
Writing $z = r\, e^{i\theta}$, we get $z^{1/2} = r^{1/2}\, e^{i\theta/2}$. As θ changes from $0 \to 2\pi$, $z^{1/2}$ changes from $r^{1/2}$ to $-r^{1/2}$. Further change from $2\pi \to 4\pi$ restores it to $r^{1/2}$. Therefore $z^{1/2}$ is double-valued over $\mathbb{C}$, with 'branches'

$$\begin{aligned} f_1(z) &= r^{1/2}\, e^{i\theta/2} \\ f_2(z) &= -r^{1/2}\, e^{i\theta/2} \end{aligned}$$



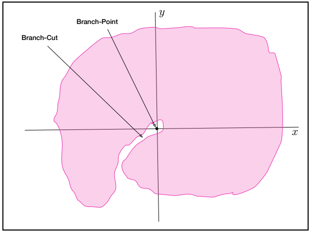
Branch-Point

As long as we do not go around the origin, the function is single-valued and analytic (Polar C.R. Equations are satisfied). The point $z = 0$ is a singular point (called 'Branch-Point') since the function is not single-valued in any region containing it.

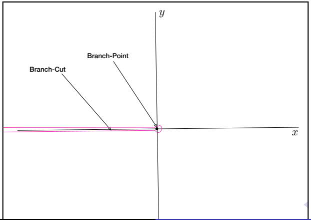


Branch-Cut

How do we maximize the region of analyticity of $f(z)$?



Convenient choice of Branch-Cut for $f(z) = z^{1/2}$



We can now define

$$z^{1/2} = r^{1/2}\, e^{i\theta/2}; \quad -\pi < \theta \leq \pi$$

Note that we can instead define

$$z^{1/2} = -r^{1/2}\, e^{i\theta/2}; \quad -\pi < \theta \leq \pi$$

Consider $f(z) = z^{1/n}$. Expressing $z = r\ e^{i(\theta+2n\pi)}$, we see that this has n branches:

$$z^{1/n} = r^{1/n}\ e^{i\theta/n}\ e^{i2\pi/n}$$

We can define a single-valued, analytic function (analytic everywhere except at the Branch-cut/Branch-Point)

$$z^{1/n} = r^{1/n}\ e^{i\theta/n}; \quad -\pi < \theta \leq \pi$$

Again, any other branch could have been taken.

Example

$f(z) = z^{1/3}$. We wish to calculate $f(-2i)$. Then we take the first branch and define

$$z^{1/3} = r^{1/3}\, e^{i\theta/3}; \quad -\pi < \theta \leq \pi$$

With this branch, $-i = e^{-i\pi/2}$. Then

$$f(-i) = e^{-i\pi/6}$$

If we were to take the 'third' branch

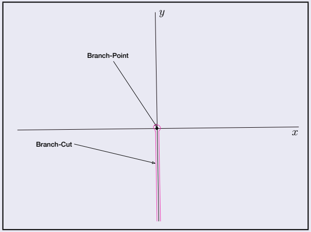
$$z^{1/3} = r^{1/3}\, e^{i\theta/3}\, e^{i2\pi/3}; \quad -\pi < \theta \leq \pi$$

then

$$\begin{aligned} f(-i) &= e^{-i\pi/6}\, e^{i2\pi/3} \\ &= e^{i\pi/2} \\ &= i \end{aligned}$$

Problem

Define the following branch-cut for the multifunction $f(z) = z^{1/n}$



Taking the first branch of $f(z) = z^{1/3}$*, evaluate* $f(-1-i)$.

Analyticity of z^a; $a \in \mathbb{R}$

Let $f(z) = z^a$, where a is a rational number. Clearly, it will have a finite number of branches, given by

$$z^a = r^a \; e^{ia\theta} \; e^{i2n\pi a}$$

Writing $a = p/q$, it can be seen that there will be q branches. If a is irrational, there will be an infinite number of branches. Within any one branch, expressing $z^a = u(r,\theta) + iv(r,\theta)$, we see that

$$\begin{aligned} u(r,\theta) &= r^a \cos\left[a\left(\theta + 2n\pi\right)\right] \\ v(r,\theta) &= r^a \sin\left[a\left(\theta + 2n\pi\right)\right] \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial u}{\partial r} &= a\, r^{a-1} \cos\left[a\left(\theta + 2n\pi\right)\right] \\ \frac{\partial u}{\partial \theta} &= -a\, r^a \sin\left[a\left(\theta + 2n\pi\right)\right] \\ \frac{\partial v}{\partial r} &= a\, r^{a-1} \sin\left[a\left(\theta + 2n\pi\right)\right] \\ \frac{\partial v}{\partial \theta} &= a\, r^a \cos\left[a\left(\theta + 2n\pi\right)\right] \end{aligned}$$

Then

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$
$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

which shows that z^a is analytic for real a. The derivative is given by

$$\begin{aligned}\frac{dz^a}{dz} &= e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)\\ &= e^{-i\theta} a r^{a-1}\left[\cos\left[a\left(\theta + 2n\pi\right)\right] + i\sin\left[a\left(\theta + 2n\pi\right)\right]\right]\\ &= e^{-i\theta} a r^{a-1}\ e^{ia(\theta+2n\pi)}\\ &= az^{a-1}\end{aligned}$$

Derivative of z^a; $a \in \mathbb{R}$

$$\frac{dz^a}{dz} = az^{a-1}$$

The complex logarithm $f(z) = \log(z)$ is defined through

$$e^{\log(z)} = z$$

Writing $\log(z) = u + iv$ and $z = |z|\ e^{i\theta}$

$$e^u\ e^{iv} = |z|\ e^{i\theta}$$

we get

$$\begin{aligned} u &= \ln|z| \\ v &= \theta + 2n\pi; \quad n \in \mathbb{Z} \end{aligned}$$

Then

$$\log(z) = \ln|z| + i(\theta + 2n\pi); \quad n \in \mathbb{Z}$$

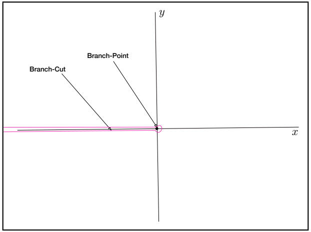
Clearly, $\log(z)$ is a multifunction with an infinit enumber of branches, labelled by n. Everytime we encircle the origin, we enter a new branch.

We can force $\log(z)$ to be single-valued by making a suitable branch cut. The 'Principal Branch' is defined by the following choice

$$Log(z) = \ln|z| + i\theta; \quad -\pi < \theta \leq \pi$$

Then all the branches are given by

$$\log(z) = Log(z) + i2n\pi$$



Example

We evaluate $Log\left(-1 - i\sqrt{3}\right)$. Clearly, $\theta = -2\pi/3$ for the Principal Branch. Furthur, $\left|-1 - i\sqrt{3}\right| = 2$. Then

$$Log\left(-1 - i\sqrt{3}\right) = \ln 2 - i\frac{2\pi}{3}$$

Given a branch, the complex logarithm is an analytic function. Writing $\log(z) = u(r, \theta) + iv(r, \theta)$, we get

$$\begin{aligned} u(r,\theta) &= \ln(r) \\ v(r,\theta) &= \theta + 2n\pi \end{aligned}$$

Then

$$\frac{\partial u}{\partial r} = \frac{1}{r}$$
$$\frac{\partial u}{\partial \theta} = 0$$
$$\frac{\partial v}{\partial r} = 0$$
$$\frac{\partial v}{\partial \theta} = 1$$

Then

$$\begin{aligned} \frac{\partial v}{\partial \theta} &= r\frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} &= -r\frac{\partial v}{\partial r} \end{aligned}$$

The derivative of the log function is

$$\begin{aligned}\frac{d\log z}{dz} &= e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)\\ &= e^{-i\theta}\frac{1}{r}\\ &= \frac{1}{z}\end{aligned}$$

Derivative of log z

$$\frac{d\log z}{dz}=\frac{1}{z}$$

Properties of Complex Logarithm

1. $$Log\left(e^{z}\right) \neq z$$

$$\begin{aligned} Log\left(e^{z}\right) &= \ln\left|e^{z}\right| + i\, Arg\left(e^{z}\right) \\ &= \ln\left(e^{x}\right) + i\, y \quad \text{Only if } -\pi < y \leq \pi \end{aligned}$$

2. $$Log\left(z_1 z_2\right) \neq Log\left(z_1\right) + Log\left(z_2\right)$$

$$\begin{aligned} Log\left(z_1 z_2\right) &= \ln\left|z_1 z_2\right| + i\, Arg\left(z_1 z_2\right) \\ &= \ln\left|z_1\right| + \ln\left|z_2\right| + i\, Arg\left(z_1\right) + i\, Arg\left(z_2\right) \end{aligned}$$

   only if $-\pi < Arg\left(z_1\right) + Arg\left(z_2\right) \leq \pi$

3. $$Log\left(\frac{z_1}{z_2}\right) \neq Log\left(z_1\right) - Log\left(z_2\right)$$

4. $$Log\left(z^{n}\right) \neq n\, Log\left(z\right); \quad \text{Equality only if } -\pi < n\, Arg(z) \leq \pi$$

Generalized Power Function

Definition:

$$z^{\alpha} = e^{\alpha \ln(z)}; \quad \alpha \in \mathbb{C}$$

Since $\log(z)$ is multivalued, so is z^{α}. A single-valued function can be defined as

$$Z^{\alpha} = e^{\alpha Log(z)}$$

with a branch cut and a branch point at $z = 0$. Properties:

1. $$Z^a Z^b = Z^{a+b}$$
$$\begin{aligned} Z^a Z^b &= e^{aLog(z)} e^{bLog(z)} \\ &= e^{(a+b)Log(z)} \\ &= Z^{a+b} \end{aligned}$$

2. $$\frac{Z^a}{Z^b} = Z^{a-b}$$

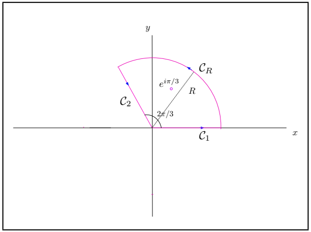
3. $$\left(Z^a\right)^b \neq Z^{ab}$$

$$I = \int_0^\infty dx\, \frac{\ln x}{x^3 + 1}$$

To evaluate this integral, we first evaluate

$$I_1 = \int_0^\infty dx\, \frac{1}{x^3 + 1}$$

Let $f(z) = 1/\left(z^3 + 1\right)$. This has poles at $z_1 = e^{i\pi/3}, z_2 = e^{i\pi}$ and $z_3 = e^{i5\pi/3}$.



Then

$$\begin{aligned}
\oint dz\, f(z) &= 2\pi i \, Res_{z=z_1} f(z) \\
&= 2\pi i \, Res_{z=z_1} \frac{1}{(z-z_1)(z-z_2)(z-z_3)} \\
&= 2\pi i \, \frac{1}{(z_1-z_2)(z_1-z_3)} \\
&= \frac{2\pi i}{3} \, e^{-i2\pi/3}
\end{aligned}$$

The integral along contour $\mathcal{C}_1$ is

$$I_{\mathcal{C}_1} = \int_0^R dr \, \frac{1}{r^3+1}$$

Along $\mathcal{C}_2, z = r\ e^{i2\pi/3};\ \ r \in [R, 0]$. Then $dz = e^{i2\pi/3} dr$ so that

$$I_{\mathcal{C}_2} = -e^{i2\pi/3} \int_0^R dr\ \frac{1}{r^3+1}$$

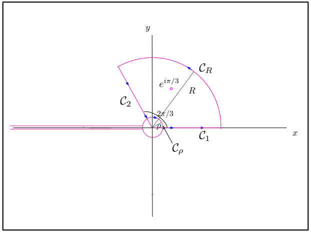
As $R \to \infty$, the integral along $\mathcal{C}_R$ goes to zero. Then,

$$\begin{aligned} \left(1 - e^{i2\pi/3}\right) \int_0^\infty dr\ \frac{1}{r^3+1} &= \frac{2\pi i}{3}\ e^{-i2\pi/3} \\ \implies I_1 &= \frac{2\pi i}{3}\ \frac{e^{-i2\pi/3}}{\left(1 - e^{i2\pi/3}\right)} \\ &= \frac{2\pi}{3\sqrt{3}} \end{aligned}$$

We now evaluate

$$I = \int_0^\infty dx \, \frac{\ln x}{x^3 + 1}$$

We choose $f(z) = \frac{Log(z)}{z^3+1}$ with a branch cut for the *Log* function, and evaluate the following closed loop contour integral



Along $\mathcal{C}_1$, $z = r$ so that

$$I_{\mathcal{C}_1} = \int_\rho^R dr \, \frac{\ln r}{r^3 + 1}$$

Along $\mathcal{C}_2, z = r\ e^{i2\pi/3}$ so that $dz = e^{i2\pi/3}dr$. Further, $Log(z) = \ln r + i2\pi/3$. Then

$$I_{\mathcal{C}_2} = -e^{i2\pi/3}\int_\rho^R dr\ \frac{\ln r}{r^3+1} - i\frac{2\pi}{3}\ e^{i2\pi/3}\int_\rho^R dr\ \frac{1}{r^3+1}$$

The integrand has poles at $z_1 = e^{i\pi/3}, z_2 = e^{i\pi}$ and $z_3 = e^{i5\pi/3}$ of which only z_1 is of interest. Then

$$\begin{aligned}
\oint dz\ f(z) &= 2\pi i\ Res_{z=z_1} f(z) \\
&= 2\pi i\ Res_{z=z_1} \frac{Log(z)}{(z-z_1)(z-z_2)(z-z_3)} \\
&= 2\pi i\ \frac{Log(z_1)}{(z_1-z_2)(z_1-z_3)} \\
&= \left(\frac{2\pi i}{3}\right)\ \frac{i\pi}{3}\ e^{-i2\pi/3} \\
&= -\frac{2\pi^2}{9}\ e^{-i2\pi/3}
\end{aligned}$$

In the limit $R \to \infty$, the integral along $\mathcal{C}_R$ goes to zero. Then substituting for $I_1 = 2\pi/3\sqrt{3}$, we finally get

$$\int_0^\infty dr\ \frac{\ln r}{r^3+1} = -\frac{2\pi^2}{27}$$

Quantum Propagation of a Relativistic Particle

In non-relativistic QM, the wavefunction of a particle at instant t if it is located at the origin (in a state of well-defined position) is given by

$$\psi(\vec{x}, t) = \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi\hbar)^3}\; e^{-iE_pt/\hbar}\; e^{i\vec{p}\cdot\vec{x}/\hbar}$$

where $E_p = \vec{p}^2/2m$ and $d^3p = dp_x dp_y dp_z$. This expression is not Lorentz invariant. To construct a Lorentz invariant expression, we conjecture that

$$\psi(\vec{x}, t) = mc^2 \int_{-\infty}^{\infty} \frac{d^3p}{(2\pi\hbar)^3\;E_p}\; e^{-iE_pt/\hbar}\; e^{i\vec{p}\cdot\vec{x}/\hbar}$$

where $E_p = \sqrt{\vec{p}^2c^2 + m^2c^4}$.

To say this is Lorentz invariant is the same as saying that under $\vec{x} \to \vec{x}'$ and $t \to t'$, ψ should remain the same, where (choosing x to be the direction along the Lorentz boost and working in units with $\hbar = c = 1$)

$$\begin{aligned} x^{0'} &= \gamma(v)\left(x^0 - vx\right) \\ x' &= \gamma(v)\left(x - vx^0\right) \\ y' &= y \\ z' &= z \end{aligned}$$

where $x^0 = ct \equiv t$. The wavefunction is

$$\psi(\vec{x}, t) = m \int_{-\infty}^{\infty} \frac{dp_x dp_y dp_z}{(2\pi\hbar)^3 \; p_0} \; e^{-i\left(p_0 x^0 - p_x x - p_y y - p_z z\right)}$$

where $p_0 = E_p = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}$.

Under $(x^0, x, y, z) \to \left(x^{0'}, x', y', z'\right)$, the wavefunction changes to

$$\begin{aligned}
\psi(\vec{x}, t) &\to m \int_{-\infty}^{\infty} \frac{dp_x\, dp_y\, dp_z}{(2\pi\hbar)^3\ p_0}\ e^{-i\left(p_0 x^{0'} - p_x x' - p_y y' - p_z z'\right)} \\
&= m \int_{-\infty}^{\infty} \frac{dp_x\, dp_y\, dp_z}{(2\pi\hbar)^3\ p_0}\ e^{-i\left(p_0\left[\gamma(v)\left(x^0 - vx\right)\right] - p_x\left[\gamma(v)\left(x - vx^0\right)\right] - p_y y - p_z z\right)} \\
&= m \int_{-\infty}^{\infty} \frac{dp_x\, dp_y\, dp_z}{(2\pi\hbar)^3\ p_0}\ e^{-i\left([\gamma(v)(p_0 + vp_x)]x^0 - [\gamma(v)(p_x + vp_0)]x - p_y y - p_z z\right)}
\end{aligned}$$

Now, we change momentum integration variables to p_x', p_y', p_z' where

$$\begin{aligned}
p_x' &= \gamma(v)\left(p_x + vp_0\right) \\
p_y' &= p_y \\
p_z' &= p_z
\end{aligned}$$

while also defining

$$p_0' = \gamma(v)\left(p_0 + vp_x\right)$$

Then it is easy to check that the exponent becomes $(p'_0 x^0 - p'_x x - p'_y y - p'_z z)$. We now need the Jacobian for the change in vaiables

$$dp'_x dp'_y dp'_z = \mathcal{J}\left(\frac{p'_x, p'_y, p'_z}{p_x, p_y, p_z}\right) dp_x dp_y dp_z$$

where

$$\mathcal{J} = \begin{vmatrix} \frac{\partial p'_x}{\partial p_x} & \frac{\partial p'_x}{\partial p_y} & \frac{\partial p'_x}{\partial p_z} \\ \frac{\partial p'_y}{\partial p_x} & \frac{\partial p'_y}{\partial p_y} & \frac{\partial p'_y}{\partial p_z} \\ \frac{\partial p'_z}{\partial p_x} & \frac{\partial p'_z}{\partial p_y} & \frac{\partial p'_z}{\partial p_z} \end{vmatrix}$$

In evaluating the partial derivatives, it should be noted that p_0 is also a function of p_x, p_y, p_z through $p_0 = \sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}$.

The Jacobian is calculated to be

$$\begin{aligned}
\mathcal{J} &= \begin{vmatrix} \gamma(v) + v\gamma(v)p_x/p_0 & v\gamma(v)p_y/p_0 & v\gamma(v)p_z/p_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \gamma(v) + v\gamma(v)p_x/p_0 \\
&= \frac{p_0'}{p_0}
\end{aligned}$$

Then

$$\begin{aligned}
dp_x' dp_y' dp_z' &= \frac{p_0'}{p_0}\, dp_x dp_y dp_z \\
\implies \frac{dp_x' dp_y' dp_z'}{p_0'} &= \frac{dp_x dp_y dp_z}{p_0}
\end{aligned}$$

Finally, the wavefunction becomes

$$\begin{aligned}\psi(\vec{x}, t) &\rightarrow m\int_{-\infty}^{\infty} \frac{dp'_x dp'_y dp'_z}{(2\pi\hbar)^3\ p'_0}\ e^{-i\left(p'_0 x^0 - p'_x x - p'_y y - p'_z z\right)} \\ &= \psi(\vec{x}, t)\end{aligned}$$

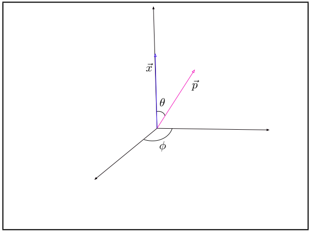
and so is invariant under a Lorentz transformation. It is easy to check that in the non-relativistic limit $c \rightarrow \infty$, the wavefunction reduces to the non-relativistic one (apart from an insignificant overall phase).

Evaluation of the Wavefunction

We now evaluate

$$\psi(\vec{x}, t) = m \int_{-\infty}^{\infty} \frac{dp_x dp_y dp_z}{(2\pi\hbar)^3 \; p_0} \; e^{-i\left(p_0 x^0 - p_x x - p_y y - p_z z\right)}$$

For a given $\vec{x}$, the set (p_x, p_y, p_z) can be visualised as a vector $\vec{p}$. For a given (p_x, p_y, p_z), this vector has length $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$ and makes angle θ with $\vec{x}$. As (p_x, p_y, p_z) change in the integral, the length p and the angle θ change



Making a change of variables from (p_x, p_y, p_z) to (p, θ, ϕ), we get

$$\begin{aligned}\psi(\vec{x}, t) &= \frac{m}{(2\pi\hbar)^3} \int_0^{2\pi} d\phi \int_0^{\infty} dp \, \frac{p^2}{\sqrt{p^2 + m^2}} \int_0^{\infty} d\theta \sin\theta \; e^{-ip_0 x^0} \; e^{ip|\vec{x}|\cos\theta} \\ &= \frac{4\pi m}{(2\pi\hbar)^3} \frac{1}{|\vec{x}|} \int_0^{\infty} dp \, \frac{p^2}{\sqrt{p^2 + m^2}} \; e^{-ip_0 x^0} \; \frac{\sin\left(p\,|\vec{x}|\right)}{p}\end{aligned}$$

We now evaluate this at space-like point $(t, \vec{x})$ such that $-t^2 + |\vec{x}|^2 > 0$. Then it is possible to choose a Lorentz frame such that $t = 0$. With this choice, we get

$$\begin{aligned}\psi &= \frac{4\pi m}{(2\pi\hbar)^3} \frac{1}{|\vec{x}|} \int_0^{\infty} dp \, \frac{p}{\sqrt{p^2 + m^2}} \; \sin\left(p\,|\vec{x}|\right) \\ &= \frac{2\pi m}{(2\pi\hbar)^3} \frac{1}{|\vec{x}|} \int_{-\infty}^{\infty} dp \, \frac{p}{\sqrt{p^2 + m^2}} \; \sin\left(p\,|\vec{x}|\right) \\ &= \frac{2\pi m}{(2\pi\hbar)^3} \frac{1}{|\vec{x}|^2} \int_{-\infty}^{\infty} du \, \frac{u}{\sqrt{u^2 + a^2}} \; \sin u\end{aligned}$$

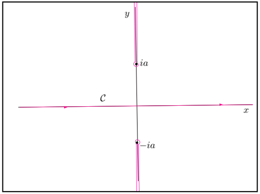
where $a = m\,|\vec{x}|$.

We now need to evaluate the integral

$$I = \int_{-\infty}^{\infty} du \, \frac{u}{\sqrt{u^2 + a^2}} \, \sin u$$

We evaluate it in the complex plane. Let

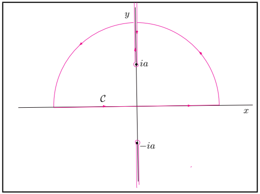
$$\begin{aligned} f(z) &= \frac{z}{\sqrt{z - ia}\,\sqrt{z + ia}} \, \sin z \\ &= \frac{1}{2i} \frac{z}{\sqrt{z - ia}\,\sqrt{z + ia}} \left(e^{iz} - e^{-iz} \right) \end{aligned}$$



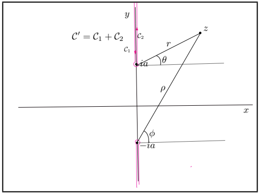
Then

$$I = \int_{\mathcal{C}} dz\; f(z)$$

We evaluate the two exponential parts separately. For the first, we consider the following contour



In the limit the radius of the semicircle approaches infinity, the contour integral reduces to



Writing $z - ia = r\ e^{i\theta}$ and $z + ia = \rho\ e^{i\phi}$ such that $-3\pi/2 < \theta \leq \pi/2,\ -\pi/2 < \theta \leq 3\pi/2$, the first exponential term contributes to the function as

$$f(z) \quad = \quad \frac{1}{2i} \frac{z\ e^{iz}}{\sqrt{z - ia}\ \sqrt{z + ia}}$$

Along $\mathcal{C}_1$, $\theta = -3\pi/2, \phi = \pi/2$ so that

$$\begin{aligned}\int_{\mathcal{C}_1} dz\, f(z) &= -\frac{1}{2}\int_{\infty}^{0} dr\, \frac{(a+r)\, e^{-(a+r)}}{r^{1/2}\rho^{1/2}} \\ &= \frac{1}{2}\int_{0}^{\infty} dr\, \frac{(a+r)\, e^{-(a+r)}}{r^{1/2}\,(2a+r)^{1/2}} \\ &= \frac{1}{2}\int_{a}^{\infty} du\, \frac{u\, e^{-u}}{\sqrt{u^2 - a^2}}\end{aligned}$$

Along $\mathcal{C}_2$, $\theta = \pi/2, \phi = \pi/2$ and it is easy to check that the integral gives the same contribution. For the second exponential term, we take a similar contour in the lower half complex plane. Again, the contribution is easily seen to be the same as for the upper half. Then, finally

$$I = 2\int_{a}^{\infty} du\, \frac{u\, e^{-u}}{\sqrt{u^2 - a^2}}$$

The wavefunction is then

$$\psi = \frac{4\pi m}{(2\pi\hbar)^3} \frac{1}{|\vec{x}|^2} \int_a^\infty du \, \frac{u \, e^{-u}}{\sqrt{u^2 - a^2}}$$

where (substituting back $\hbar$ and c),

$$a = \frac{|\vec{x}|}{(\hbar/mc)}$$

Making a simple substitution gives

$$\psi = \frac{4\pi m}{(2\pi\hbar)^3} \frac{1}{|\vec{x}|^2} \, e^{-a} \int_0^\infty ds \, \frac{(s+a) \, e^{-s}}{\sqrt{s(s+2a)}}$$

The exponential factor $e^{-a} = e^{-\frac{|\vec{x}|}{(\hbar/mc)}}$ shows that the probability of detecting the particle outside the light-cone falls exponentially with distance with a characteristic length equal to the compton wavelength $\lambda_c = \hbar/mc$ of the particle.