Laplace Transform and Applications

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Outline

An Illustration

Consider the Integral

$$I(\omega) = \int_0^\infty dt\ \sin(\omega_0 t)\ e^{-i\omega t}$$

This can be thought of as a Fourier Transform of the function $f(t) = \theta(t)\sin(\omega_0 t)$. To make sense of this integral, we evaluate it as

$$I(\omega) = \lim_{\epsilon \to 0+} \int_0^\infty dt\ \sin(\omega_0 t)\ e^{-(\epsilon + i\omega)t}$$

This is just an analytical continuation of the function

$$I(z) = \int_0^\infty dt\ \sin(\omega_0 t)\ e^{-zt}$$

to the imaginary axis. $I(z)$ is analytic for $Re(z) > 0$. Expanding the sin in terms of exponentials and evaluating integrals as antiderivatives gives (for $Re(z) > 0$)

$$I(z) = \frac{\omega_0}{z^2 + \omega_0^2}$$

Then

$$I(\omega) = \frac{\omega_0}{\omega_0^2 - \omega^2}$$

Definition

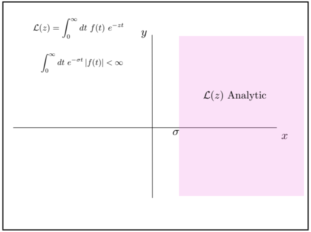
Laplace transform of complex function $f(t)$:

$$\mathcal{L}(s) = \int_0^\infty dt\ f(t)\ e^{-st}; \quad Re(s) \geq \sigma$$

where σ is such that

$$\int_0^\infty dt\ e^{-\sigma t}\,|f(t)| < \infty$$

$\mathcal{L}(s)$ is analytic for $Re(s) > \sigma$.



Relation with Fourier Transform

Consider the Fourier transform of a tim-varying signal $f(t)$ which is switched on at a finite time T_0. Then

$$\tilde{f}(\omega) = \int_{T_0}^{\infty} dt\ f(t)\ e^{-i\omega t}$$

Any physical signal will die down as $t \to \infty$. Shifting the t integral gives

$$\tilde{f}(\omega) = e^{-i\omega T_0} \int_{0}^{\infty} dt\ f(t)\ e^{-i\omega t}$$

This is a 'half-interval' Fourier Transform. Then fourier transform of any physical signal can be reduced to a half-integral transform.

Half-Interval Fourier Transform

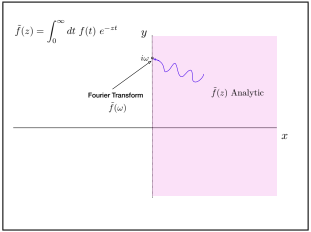
Let

$$\tilde{f}(\omega) = \int_0^\infty dt\; f(t)\; e^{-i\omega t}$$

where it is assumed that this integral exists. We generalise to the complex plane

$$\tilde{f}(z) = \int_0^\infty dt\; f(t)\; e^{-zt}; \quad Re(z) \geq 0$$

where clearly this is a Laplace Transform. $\tilde{f}(z)$ is analytic for $Re(z) > 0$. Then $\tilde{f}(\omega)$ is the analytic continuation of $\tilde{f}(z)$ to the imaginary axis.



$$\mathcal{L}(\alpha + i\omega) = \int_0^\infty dt\ f(t)\ e^{-\alpha t}\ e^{-i\omega t}; \quad \alpha \geq \sigma$$

Extend the definition of $f(t)$ such that $f(t) = 0$ for $t < 0$. Then

$$\mathcal{L}(\alpha + i\omega) = \int_{-\infty}^\infty dt\ f(t)\ e^{-\alpha t}\ e^{-i\omega t}$$

This can be visualised as a Fourier transform of $f(t)\ e^{-\alpha t}$. Using inverse Fourier transform we get

$$f(t)\ e^{-\alpha t} = \int_{-\infty}^\infty \frac{d\omega}{2\pi}\ \mathcal{L}(\alpha + i\omega)\ e^{i\omega t}$$

which gives

$$\begin{aligned} f(t) &= \int_{-\infty}^\infty \frac{d\omega}{2\pi}\ \mathcal{L}(\alpha + i\omega)\ e^{(\alpha + i\omega)t} \\ &= \int_{\mathcal{C}} \frac{dz}{2\pi i}\ \mathcal{L}(z)\ e^{zt} \end{aligned}$$

where $\mathcal{C}$ is a contour parallel to the imaginary axis and shifted by α.

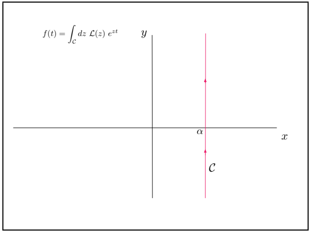
Summary

Laplace Transform

$$\mathcal{L}(s) = \int_0^\infty dt\ f(t)\ e^{-st}; \quad Re(s) \geq \sigma$$

Inverse Laplace Transform

$$f(t) = \int_{\mathcal{C}} \frac{dz}{2\pi i}\ \mathcal{L}(z)\ e^{zt}$$



Damped Harmonic Oscillator

$$m\,\frac{d^2x(t)}{dt^2} + b\,\frac{dx(t)}{dt} + k\,x(t) = 0; \quad x(0) = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = 0$$

We take Laplace Transform of both sides

$$m\,\mathcal{L}\left[\frac{d^2x(t)}{dt^2}\right] + b\,\mathcal{L}\left[\frac{dx(t)}{dt}\right] + k\,\mathcal{L}\left[x(t)\right] = 0$$

$$\begin{aligned}\mathcal{L}\left[\frac{dx(t)}{dt}\right] &= \int_0^\infty dt\,\frac{dx(t)}{dt}\,e^{-st} \\ &= \int_0^\infty dt\left[\frac{d}{dt}\left(x(t)\,e^{-st}\right) + s\,x(t)\,e^{-st}\right] \\ &= -x_0 + s\,\mathcal{L}\left[x(t)\right]\end{aligned}$$

Similarly

$$\mathcal{L}\left[\frac{d^2x(t)}{dt^2}\right] = -s\,x_0 + s^2\,\mathcal{L}\left[x(t)\right]$$

Substituting in the D.E gives

$$\begin{aligned}\mathcal{L}[x(t)] &= x_0 \frac{ms+b}{ms^2+bs+k}\\ &= x_0 \frac{s+b/m}{(s+b/2m)^2+\omega_1^2}\\ &= x_0 \frac{s+b/2m}{(s+b/2m)^2+\omega_1^2} + x_0 \frac{b/2m}{(s+b/2m)^2+\omega_1^2}\end{aligned}$$

where $\omega_1 = \sqrt{k/m - b^2/4m^2}$. We now need to take the inverse transform. Observation: Each term is the Laplace transform of some function shifted by $b/2m$.

Property: If $\mathcal{L}(s)$ is the Laplace transform of $f(t)$ then $\mathcal{L}(s+a)$ is the Laplace Transform of $f(t)\, e^{-at}$. Then we just need to determine Inv. Lap. Trans. of $s/(s^2+\omega_1^2)$ and $1/(s^2+\omega_1^2)$.

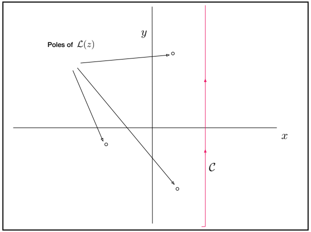
Property: If $\mathcal{L}(s)$ is the Lap. Trans. of $f(t)$ then

$$s\,\mathcal{L}(s) = f(0) + \mathcal{L}\left[\frac{df(t)}{dt}\right]$$

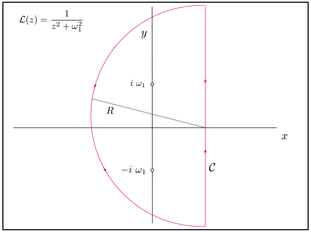
Therefore we just need to evaluate the Inv. Lap. Trans. of $\mathcal{L}(s) = 1/(s^2 + \omega_1^2)$.
Recall:

$$f(t) = \int_{\mathcal{C}} \frac{dz}{2\pi i}\,\mathcal{L}(z)\,e^{zt}$$

where $\mathcal{C}$ is a contour such that all the poles of $\mathcal{L}(z)$ are to the left of the contour



Here $\mathcal{L}(z) = 1/(z^2 + \omega_1^2)$ with poles at $z = \pm i\omega_1$.



$$f(t) = \int_{\mathcal{C}} \frac{dz}{2\pi i}\, \mathcal{L}(z)\, e^{zt} = \lim_{R\to\infty} \oint \frac{dz}{2\pi i}\, \mathcal{L}(z)\, e^{zt} = \frac{\sin(\omega_1 t)}{\omega_1}$$

where the contribution form the semi-circular part of the contour vanishes as $R \to \infty$.

Then

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left[\mathcal{L}\left[x(t)\right]\right] \\ &= x_0\,\mathcal{L}^{-1}\left[\frac{s+b/2m}{(s+b/2m)^2+\omega_1^2}\right] + x_0\,\frac{b}{2m}\mathcal{L}^{-1}\left[\frac{1}{(s+b/2m)^2+\omega_1^2}\right] \\ &= x_0\,e^{-bt/2m}\frac{df(t)}{dt} + x_0\,\frac{b}{2m}\,e^{-bt/2m}\,f(t) \\ &= x_0\,e^{-bt/2m}\left[\cos\left(\omega_1 t\right) + \frac{b}{2m\omega_1}\,\sin\left(\omega_1 t\right)\right] \end{aligned}$$