Motivation for Complex Numbers

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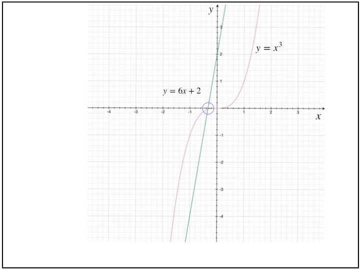
Outline

Solution to Cubic Equations

$$x^3 = 3px + 2q$$

A solution will always exist.



Algebraic solution:

$$x = \left[q + \left(q^2 - p^3\right)^{1/2}\right]^{1/3} + \left[q - \left(q^2 - p^3\right)^{1/2}\right]^{1/3}$$

$p = 5, q = 2$

$$x^3 = 15x + 4$$

$$x = \left[2 + (-121)^{1/2}\right]^{1/3} + \left[2 - (-121)^{1/2}\right]^{1/3}$$

$x = 4$ is a solution.

Conjecture: $\sqrt{-121} = 11i$

$$x = [2 + 11i]^{1/3} + [2 - 11i]^{1/3}$$

Conjecture: $\sqrt{2+11i} = 2 + ai$, $\sqrt{2-11i} = 2 - ai$, $a \in \mathbb{R}$
Assuming 'intuitive' algebraic rules (Closure, Commutative, Associative, Distributive properties), conjecture consistent with $a = \pm 1$

Complex Algebra

Complex number: $z = x + iy \quad x, y \in \mathbb{R}$
Algebraic rules for addition and multiplication:
$z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

$$z_1.z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)$$

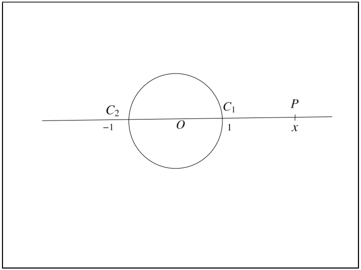
Factorization

Factorization of $x^n - 1$

$$
\begin{aligned}
x^2 - 1 &= (x+1)(x-1) \\
x^3 - 1 &= (x-1)(x^2+x+1) \\
x^4 - 1 &= (x-1)(x+1)(x^2+1)
\end{aligned}
$$

Visual Factorization

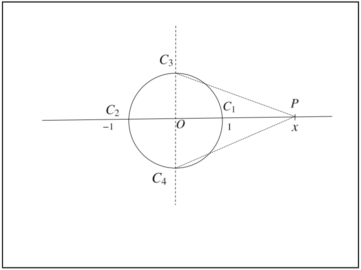
$$x^2 - 1 = (x+1)(x-1)$$



$$x^2 - 1 = (PC_1)\,(PC_2)$$

Visual Factorization of $x^4 - 1$

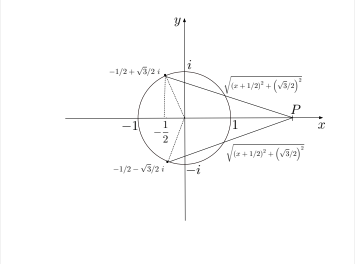
$$x^4 - 1 = (x-1)(x+1)\sqrt{x^2+1}\sqrt{x^2+1}$$



$$x^4 - 1 = (PC_1)\,(PC_2)\,(PC_3)\,(PC_4)$$

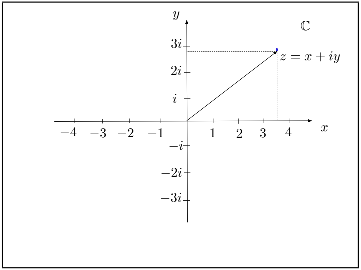
Visual factorization of $x^3 - 1$

$$\begin{aligned} x^3 - 1 &= (x-1)(x^2 + x + 1) \\ &= (x-1)\sqrt{(x+1/2)^2 + \left(\sqrt{3}/2\right)^2}\sqrt{(x+1/2)^2 + \left(\sqrt{3}/2\right)^2} \end{aligned}$$

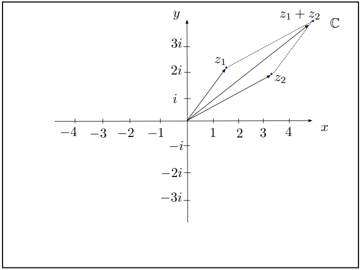


The Complex Plane

Complex numbers as ‘vectors’



Visualization of addition of complex numbers: Parallelogram Rule



Polar Representation

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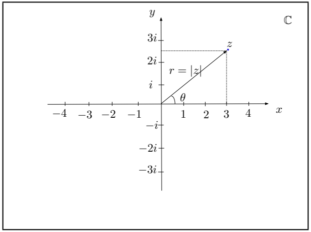

$$z = r(\cos\theta + i\sin\theta)$$

Euler's formula

Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + ... \quad \text{(definition)} \\ &= \left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 ..\right) + i\left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 ..\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Given series definition of e^a, it follows

$$e^a e^b = e^{a+b} \quad a, b \in \mathbb{C}$$

$$\begin{aligned} z_1.z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= (r_1 r_2)\, e^{i(\theta_1+\theta_2)} \end{aligned}$$

Visualization of product of complex numbers: