Classical Mechanics

A. Gupta

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Outline

References

References

- ‘An Introduction to Mechanics’ by D. Kleppner and R. Kolenkow (mostly for Problems)
- The Feynman Lectures in Physics - Volume 1
- ‘Introduction to Classical Mechanics’ by David Morin (for Problems, discussion on approximations and dimensional analysis)
- Internet Resource 1: ‘Fundamentals of Physics with Rammurthy Shankar’ - by YaleCourses (YouTube Lectures)
- ‘Classical Mechanics’ - Leonard Susskind, Stanford University (YouTube Lectures)

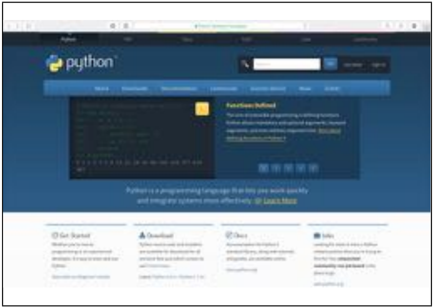
Computational Physics

Exploring Physics on the computer: Simulating mechanical systems

My programming language of choice: **Python**

Link: **https://www.python.org**

Download and install soon as possible.



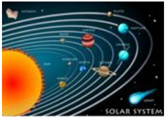
Final Instructions

- Switch off mobile phones before entering the class.
- No conversation in class.
- Feel free to stop me and ask questions.
- **Do not waste your precious time and the money the taxpayer shells out to subsidise your education.** Start working day one, First Semester can make or break you.
- Feel free to contact me with questions at: **abhinav.gupta@ststephens.edu**
- Create a shared Dropbox Folder for Lecture Notes/Presentations. You should aim to rely mostly on the lectures, supplemented with problems from references.
- There will be a Tutorial class once a wekk for problem solving. Form groups of 4-5 students for Tutorials. Once a week, one group meets me personally in my office to discuss possible issues/problems. Arrange to sit as a group, if possible, in class.

Are natural phenomena predictable?

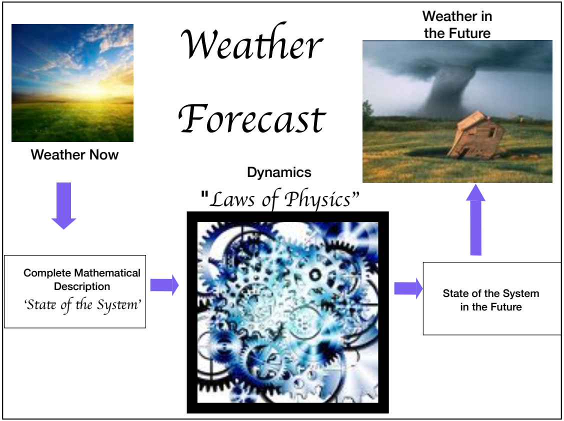
Evidence for underlying Laws:

- Cyclicity of natural phenomena



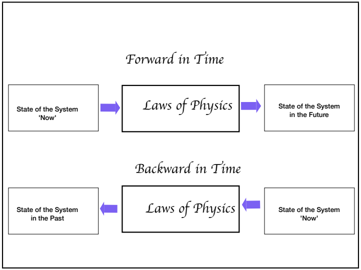
- Predicable trajectories: Sports!

Predicting the Future



Assumptions:

1. Nature is not arbitrary. There exist fundamental Laws that govern its behaviour.
2. Given initial data, these Laws allow us to (in principle) predict the future to arbitrary accuracy.
3. The Laws are time-reversible. Given present data, we can find information about the past to arbitrary accuracy*.



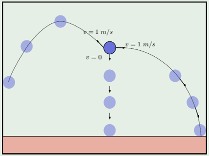
Position and Velocity

Definition

State of a System: Minimum information about the system which allows us to predict the same information at a later time (or a time in the past)

Example

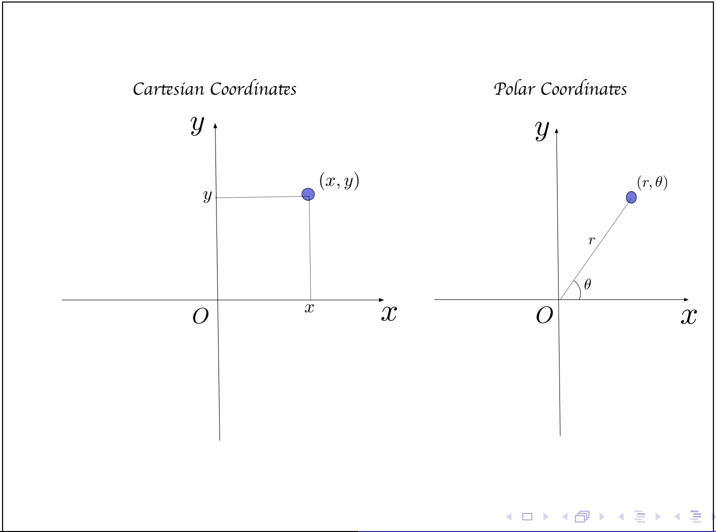
Object falling near the surface of the Earth
Specifying the initial position and *velocity* uniquely determines motion.



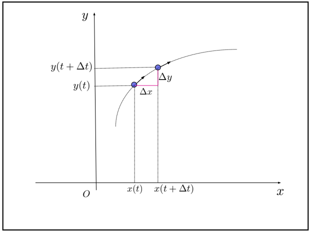
Conjecture

The state of an object is completely specified by its position and velocity.

Labelling Position



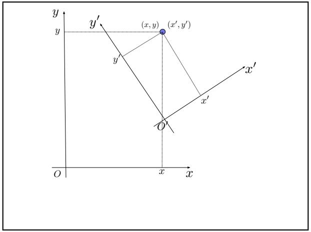
Velocity



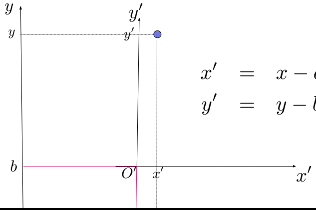
'How Fast': Change in position coordinates in unit time.
Instantaneous measure of 'How Fast': Instantaneous velocity.

$$
\begin{aligned}
v_x &= {}_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \\
&= \frac{dx(t)}{dt} \\
v_y &= {}_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t} \\
&= \frac{dy(t)}{dt}
\end{aligned}
$$

Coordinate Transformations

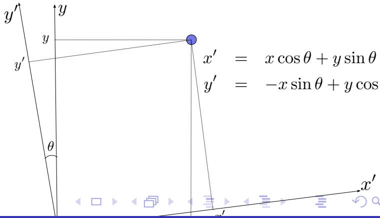


Translation



$$x' = x - a$$
$$y' = y - b$$

Rotation

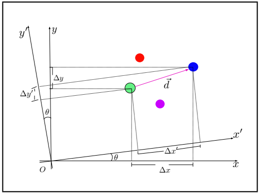


$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

Vectors

Coordinate systems versus 'absolute' quantities

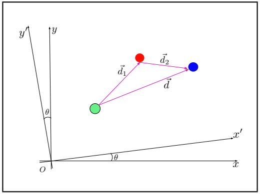
Position of an object relative to other objects (Displacement)



$$\begin{aligned}\Delta x' &= \Delta x\cos\theta + \Delta y\sin\theta \\ \Delta y' &= -\Delta x\sin\theta + \Delta y\cos\theta\end{aligned}$$

$$\begin{aligned}\vec{d} &\equiv (\Delta x, \Delta y) \\ &\equiv (\Delta x', \Delta y')'\end{aligned}$$

Algebra of displacements:



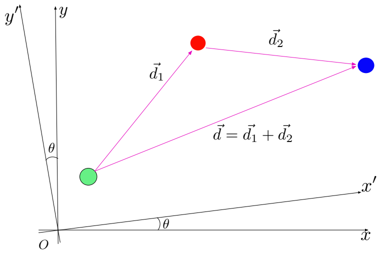
We note:
If $\vec{d}_1 \equiv (\Delta x_1, \Delta y_1)$ and $\vec{d}_2 \equiv (\Delta x_2, \Delta y_2)$ then $\vec{d} \equiv (\Delta x_1 + \Delta x_2, \Delta y_1 + \Delta y_2)$.
Similarly
If $\vec{d}_1 \equiv (\Delta x_1', \Delta y_1')'$ and $\vec{d}_2 \equiv (\Delta x_2', \Delta y_2')'$ then $\vec{d} \equiv (\Delta x_1' + \Delta x_2', \Delta y_1' + \Delta y_2')'$.
This suggests we define an operation of **'addition'**.

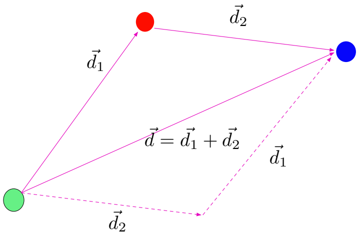
Addition of displacements:

$$\begin{aligned}(\Delta x_1, \Delta y_1) + (\Delta x_2, \Delta y_2) &= (\Delta x_1 + \Delta x_2, \Delta y_1 + \Delta y_2) \\ (\Delta x_1', \Delta y_1')' + (\Delta x_2', \Delta y_2')' &= (\Delta x_1' + \Delta x_2', \Delta y_1' + \Delta y_2')'\end{aligned}$$

Both equations have identical geometrical content:

$$\vec{d}_1 + \vec{d}_2 = \vec{d}$$

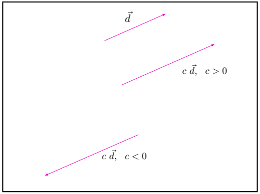




Multiplication of displacement with a number:
Define

$$c\,(\Delta x, \Delta y) = (c\Delta x, c\Delta y)$$

Geometrical Interpretation:



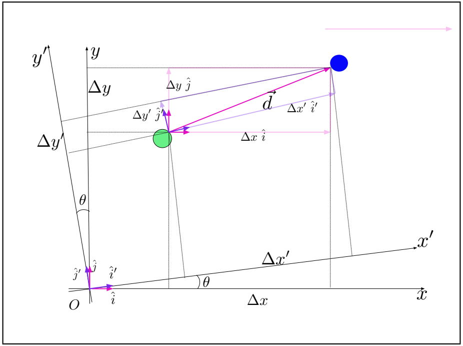
Unit Displacements

$$\begin{aligned} (\Delta x, \Delta y) &= \Delta x\,(1,0) + \Delta y\,(0,1) \\ \Longrightarrow \vec{d} &= \Delta x\,\hat{i} + \Delta y\,\hat{j} \end{aligned}$$

Similarly

$$\begin{aligned} (\Delta x', \Delta y')' &= \Delta x'\,(1,0)' + \Delta y'\,(0,1)' \\ \Longrightarrow \vec{d} &= \Delta x'\,\hat{i}' + \Delta y\,\hat{j}' \end{aligned}$$

where $\hat{i} \equiv (1,0)$, $\hat{j} \equiv (0,1)$, $\hat{i}' \equiv (1,0)'$, $\hat{j}' \equiv (0,1)'$ are displacements of unit length.



$$\vec{d} = \Delta x\,\hat{i} + \Delta y\,\hat{j}$$
$$= \Delta x'\,\hat{i}' + \Delta y\,\hat{j}'$$

Vectors

Definition

A ‘Vector’ $\vec{A}$ (in a plane) is any physical quantity that in any given Cartesian coordinate system (x, y) is represented by a pair of numbers (A_x, A_y) such that the pair of numbers (A_x, A_y) and (A'_x, A'_y) assigned in two different Cartesian systems (x, y) and (x', y') respectively oriented by an angle θ (with origins displaced by an arbitrary amount) are related as

$$\begin{aligned} A'_x &= A_x \cos\theta + A_y \sin\theta \\ A'_y &= -A_x \sin\theta + A_y \cos\theta \end{aligned}$$

Note

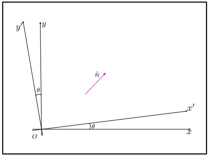
This is the same relation as between components of a displacement in space.

How do we know that the different pairs of numbers (A_x, A_y) and (A'_x, A'_y) represent the *same* quantity?

- We can construct a number from each pair which will be agreed upon in all coordinate systems. This number is called the *magnitude* of the vector $\vec{A}$, denoted by $\left|\vec{A}\right|$

$$\left|\vec{A}\right| = \sqrt{A_x^2 + A_y^2} = \sqrt{A_x'^2 + A_y'^2}$$

- Consider a unit dispacement in space (a direction, or a 'unit vector') $\hat{n}$ with components (n_x, n_y) and $(n'_x, n'_y)'$ in coordinate systems (x, y) and (x', y').

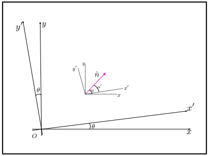


We can construct another number which will be agreed upon in both coordinate systems

$$A_x n_x + A_y n_y = A'_x n'_x + A'_y n'_y$$

We denote this number as $\vec{A} \cdot \hat{n}$.

Since $A_x^2 + A_y^2 = \left|\vec{A}\right|^2$, we can define an angle ϕ such that $A_x = \left|\vec{A}\right| \cos\phi$ and $A_y = \left|\vec{A}\right| \sin\phi$. Similarly, we can define angle ϕ' such that $A'_x = \left|\vec{A}\right| \cos\phi'$ and $A_y = \left|\vec{A}\right| \sin\phi'$. What is the significance of these angles?



$n_x = \cos\alpha$, $n_y = \sin\alpha$, $n'_x = \cos\alpha'$, $n'_y = \sin\alpha'$, with $\alpha - \alpha' = \theta$.

$$
\begin{aligned}
\frac{\vec{A}\cdot\hat{n}}{\left|\vec{A}\right|} &= \frac{A_x n_x + A_y n_y}{\left|\vec{A}\right|} \\
&= \frac{\left|\vec{A}\right| \cos\phi\cos\alpha + \left|\vec{A}\right| \sin\phi\sin\alpha}{\left|\vec{A}\right|} \\
&= \cos(\phi - \alpha)
\end{aligned}
$$

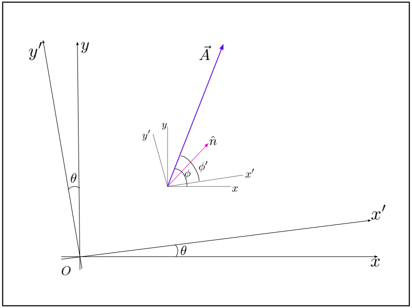
However,

$$\begin{aligned}\frac{\vec{A}\cdot\hat{n}}{\left|\vec{A}\right|} &= \frac{A'_x n'_x + A'_y n'_y}{\left|\vec{A}\right|} \\ &= \frac{\left|\vec{A}\right|\cos\phi'\cos\alpha' + \left|\vec{A}\right|\sin\phi'\sin\alpha'}{\left|\vec{A}\right|} \\ &= \cos(\phi' - \alpha')\end{aligned}$$

Then

$$\begin{aligned}\phi' - \alpha' &= \phi - \alpha \\ \implies \phi - \phi' &= \alpha - \alpha' \\ &= \theta\end{aligned}$$

This allows us to sketch $\vec{A}$ as a directed 'arrow' with length $\left|\vec{A}\right|$ and direction inclined by ϕ relative to x-axis and ϕ' relatve to the x'-axis



Vector Algebra

Since any $\vec{A}$ mimics properties of displacement, we can geometrically define addition of vectors according to the parallelogram law for displacements. It then follows that vectors can be multiplied by scalars (numbers) and added to each other with properties

1. $c\vec{A}$ has the same/opposite direction as $\vec{A}$ and magnitude equal to $|c|\,\left|\vec{A}\right|$ if c is positive/negative.
2. $\vec{A}+\vec{B}=\vec{B}+\vec{A}$
3. $\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$
4. $c\left(\vec{A}+\vec{B}\right)=c\vec{A}+c\vec{B}$

This allows us to write

$$\begin{aligned}
\vec{A} &= A_x\,\hat{i}+A_y\,\hat{j} \\
&= A'_x\,\hat{i}'+A'_y\,\hat{j}' \\
c\,\vec{A} &= (c\,A_x)\,\hat{i}+(c\,A_y)\,\hat{j} \\
\vec{A}+\vec{B} &= (A_x+A_y)\,\hat{i}+(B_x+B_y)\,\hat{j}
\end{aligned}$$

Scalar Product

A **Scalar** quantity is a number which is measured to be the same in all coordinate systems.

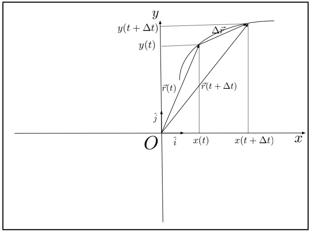
Given two vector quantities $\vec{A}$ and $\vec{B}$, we can construct a **scalar product** defined in any one coordinate system as

$$\begin{aligned} \vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y \\ &= A'_x B'_x + A'_y B'_y \end{aligned}$$

Scalar quantities are of fundamental importance in Physics since **all measurements are numbers and so must be scalars**.

Velocity as a Vector

Position vector: Displacement vector relative to some fixed point (which may or may not be chosen to be the origin of some coordinate system).



With a coordinate system attached, we can write

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$$

Then, the displacement vector from t to $t + \Delta t$ will be

$$\begin{aligned}\Delta\vec{r} &= \vec{r}(t+\Delta t) - \vec{r}(t) \\ &= (x(t+\Delta t) - x(t))\,\hat{i} + (y(t+\Delta t) - y(t))\,\hat{j}\end{aligned}$$

Multiplying with $(1/\Delta t)$

$$\frac{\Delta\vec{r}}{\Delta t} = \left(\frac{x(t+\Delta t) - x(t)}{\Delta t}\right)\hat{i} + \left(\frac{y(t+\Delta t) - y(t)}{\Delta t}\right)\hat{j}$$

Taking the limit $\Delta t \to 0$, we get the velocity vector

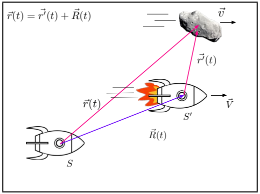
$$\begin{aligned} \vec{v} &= \lim_{\Delta t\to 0} \frac{\Delta\vec{r}}{\Delta t} \\ &= \lim_{\Delta t\to 0} \left(\frac{x(t+\Delta t) - x(t)}{\Delta t}\right)\hat{i} + \left(\frac{y(t+\Delta t) - y(t)}{\Delta t}\right)\hat{j} \\ &= v_x\,\hat{i} + v_y\,\hat{j} \end{aligned}$$

where $v_x = dx/dt$ and $v_y = dy/dt$. It is easy to check that under a coordinate transformation, the velocity components transform as

$$\begin{aligned} v'_x &= v_x \cos\theta + v_y \sin\theta \\ v'_y &= -v_x \sin\theta + v_y \cos\theta \end{aligned}$$

which they ought to, since they describe a vector quantity.

Vector Addition of Velocity



Let velocity of object observed by S' be $\vec{v}'$ and that observed by S be $\vec{v}$. Then since

$$\vec{r}(t) = \vec{R}(t) + \vec{r'}(t)$$

Therefore

$$\vec{r}(t+\Delta t) - \vec{r}(t) = \vec{R}(t+\Delta t) - \vec{R}(t) \ + \ \vec{r'}(t+\Delta t) - \vec{r'}(t)$$

Multiplying with $1/\Delta t$ and taking limit $\Delta t \to 0$

Relative Velocity

$$\vec{v} = \vec{V} + \vec{v}'$$

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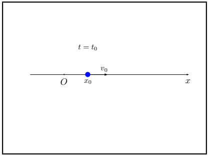
Outline

Predicting the Future

Object constrained to move along a line



Given position and velocity at one instant, what information do we need to predict these at other instants?

Observation

We only need to be able to find an algorithm that determines the position and velocity at an infinitesimally close instant, given these at one instant. This would allow us to find the position and velocity recursively.

Information about position and velocity at instant t allows us to determine the new position at instant $t + \Delta t$

$$x(t + \Delta t) \approx x(t) + v(t)\Delta t$$

This follows from the fact that velocity is rate of change of position.
What do we need to know in addition at instant t to determine the velocity at $t + \Delta t$?
We need to know the rate of change of velocity at instant t (acceleration)

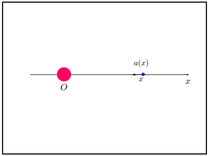
$$a(t) = \frac{dv}{dt}$$

Given this, we can find the new velocity

$$v(t + \Delta t) \approx v(t) + a(t)\Delta t$$

We can continue this recursively, if we know the acceleration at all instants of time.
However, **we *do not* know the acceleration as a function of time beforehand, since we do not know the motion of the object beforehand!**

The way out:
What if we were to know the acceleration of the object, if we know its position and velocity? That is, $a = a(x, v)$.
Example: A particle with charge in the vicinity of another (much more massive) particle with opposite charge, which attracts it. Experimentally, it is observed that the closer the particle to the attractor, the more its acceleration towards it. We can empirically determine the acceleration of the particle as a function of the distance from the attractor



Given $a = a(x)$, we can set up an algorithm.

The Algorithm

We are given position and veocity at instant t, and the acceleration as a function of position. Then, we know the acceleration at instant t (since we know the position at this instant). Therefore,

$$\begin{aligned} x(t+\Delta t) &\approx x(t)+v(t)\Delta t \\ v(t+\Delta t) &\approx v(t)+a\left(x(t)\right)\Delta t \end{aligned}$$

The position at $t+\Delta t$ allows us to calculate the acceleration at this instant, $a = a\left(x(t+\Delta t)\right)$. Then, we can compute the position and velocity at $t+2\Delta t$

$$\begin{aligned} x(t+2\Delta t) &\approx x(t+\Delta t)+v(t+\Delta t)\Delta t \\ v(t+2\Delta t) &\approx v(t+\Delta t)+a\left(x(t+\Delta t)\right)\Delta t \end{aligned}$$

and so on.

Uniform Acceleration

Problem

Consider an object moving with uniform acceleration. Given its position and velocity at $t = 0$, we wish to determine them at some instant T. We first divide the total duration of motion into N parts of duration Δt, such that $T = N\Delta t$. Then, given that the acceleration is a constant a independent of position, we can set up the iterative algorithm which allows us to 'hop' from instant $t_{n-1} = (n-1)\Delta t$ to the instant $t_n = n\Delta t$. The algorithm will be

$$\begin{aligned} x(n\Delta t) - x((n-1)\Delta t) &= v((n-1)\Delta t)\ \Delta t \\ v(n\Delta t) - v((n-1)\Delta t) &= a\,\Delta t \end{aligned}$$

Show that this algorithm gives $v(n\Delta t) = v(0) + (n\Delta t) \times a$ and

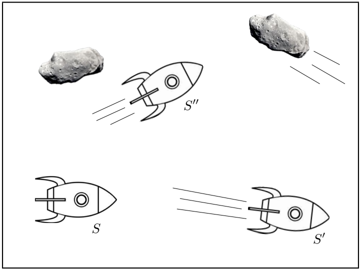
$$x(n\Delta t) = x(0) + v(0) \times (n\Delta t) + a(\Delta t)^2 \times \frac{n(n-1)}{2}$$

Using these expressions, evaluate $x(T)$ and $v(T)$ by taking the limit $N \to \infty$ and $\Delta t \to 0$ with $N\Delta t \to T$ to get

$$\begin{aligned} v(T) &= v(0) + a\,T \\ x(T) &= x(0) + v(0)\ T + \frac{1}{2} a\,T^2 \end{aligned}$$

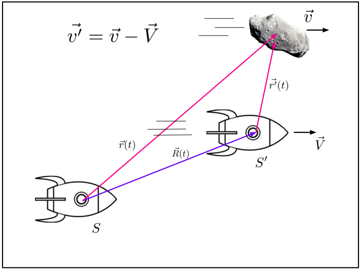
Observers in Deep Space

We consider a set of observers in spacecrafts in deep space, far from any gravitating objects. We assume they are not firing their rockets and are in uniform relative motion. From the point of view of these observers, 'absolute state of rest' is meaningless, since if something is rest with respect to one, it is not with respect to another observer. Clearly, all observers are equivalent, there is no one special observer (since there is nothing else in the environment to single out his motion, unlike say on the surface of the Earth). These observers set out to discover the Laws of Physics, which should be the same for all of them, since they are indistinguishable



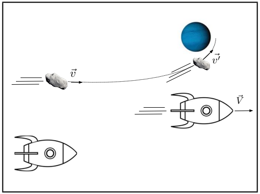
Inertial Frames

Observation: Floating asteroids and other debris in space move with uniform velocity, unless they hit each other or something else. Can this be a 'Law'? It is possible, *provided all such observers conclude the same thing.*



Since $\vec{V}$ does not change with time (no rockets fired), if $\vec{v}$ is constant, so it $\vec{v'}$. Then this 'Law' will be true for all observers in spacecrafts which are not firing their rockets.

Change in Motion



Observation: Floating asteroids and other debris in space move with uniform velocity, unless they hit each other or something else, *or are in the vicinity of a planet*. Can this be a 'Law'?

$$\begin{aligned} \vec{v'}(t) &= \vec{v}(t) - \vec{V} \\ \Longrightarrow \frac{d\vec{v'}}{dt} &= \frac{d\vec{v}}{dt} + 0 \;\; (\text{since } \tfrac{d\vec{V}}{dt} = 0) \\ \Longrightarrow \vec{a'} &= \vec{a} \end{aligned}$$

Observers in uniform relative motion measure acceleration of objects to be the same. Then, this can be a 'Law'.

The First Law of Motion

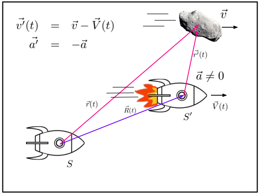
First Law:

First Law: There exist observers (such as aboard freely floating spacecrafts in deep space) who conclude that objects move with uniform motion unless an *influence* acts on them (such as other objects hitting them, they being pulled by planets, etc.)

Note: Here, influence involves other objects in the environment of the 'test' objects, such as other objects, planets, etc.
Such 'frames of reference' are called **Inertial Frames**. Then, the first law is basically a statements saying that there *exist* such frames.
Note: Given an inertial frame, any other in uniform motion relative to it is an inertial frame. Therefore, there are an *infinite number of inertial frames*.

Non-Inertial Frames

What about an observer accelerating relative to an inertial observer?



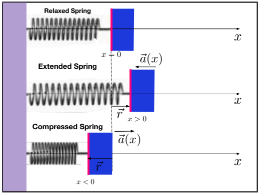
$$\begin{aligned} \vec{v'}(t) &= \vec{v} - \vec{V}(t) \\ \Longrightarrow \frac{d\vec{v'}}{dt} &= -\frac{d\vec{V}}{dt} \quad (\text{since } \frac{d\vec{v}}{dt} = 0\ ) \\ \Longrightarrow \vec{a'} &= -\vec{a} \end{aligned}$$

Observer accelerating relative to inertial observer will conclude that objects can accelerate without any external influence. Therefore, *The First Law is not valid for observers accelerating relative to inertial observers*. These frames of reference frames are called **Non-Inertial Frames of Reference**.

Interactions and Acceleration

In inertial frames, objects interacting with other objects will accelerate. Can we find a Law that predicts how this acceleration depends on this interaction?

Experiments with Springs

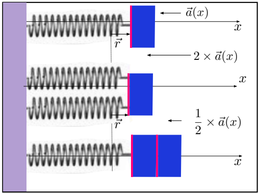


Observations

- Acceleration of the object depends upon its position. It is directed opposite to displacement of the object from 'relaxed' position. For small displacements, it is found that
$$\vec{a} \propto -\vec{r}$$
- The 'external influence' in this case is the spring. The magnitude of acceleration for a given displacement depends on the nature of the spring, and we term it *strength* of the influence.

Force

Two springs:



Observations

- Two springs (for the same displacement) lead to twice the acceleration compared with that due to one spring. In general, N springs lead to N times the acceleration due to a single spring.
- If two objects are lumped into one, the acceleration (everything else being same) is halved. In general, if N such objects are lumped, acceleration is $1/N$ times that of one object.

The Second Law (for springs)

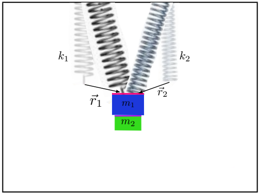
We can summarise the observations in the form of an equation

$$\vec{F} = m\,\vec{a}$$

where $\vec{F}$ quantifies the 'influence' of the springs, which we term as *Force* due to the springs. For a single spring and small displacement (of the object), $\vec{F} = -k\,\vec{r}$ where constant k is a measure of the *strength* of the force (which depends on the nature of the spring). For N (identical) springs, $\vec{F} = -N \times k\,\vec{r}$. The constant m is a property of the object(s) on which the force acts. It is proportional to the number of (identical) objects lumped together. We term this constant as *mass*.

We can use this equation to *calibrate* masses of objects by choosing some standard object to have a 'unit' mass. For a given force (spring), accelerations of different objects will be in inverse ratio of their masses.

What if we couple different kinds of springs, and different kinds of objects?



We still get

$$\vec{F} = m\,\vec{a}$$

where now

$$\begin{aligned}\vec{F} &= -k_1\vec{r}_1 - k_2\vec{r}_2 \\ &= \vec{F}_1 + \vec{F}_2\end{aligned}$$

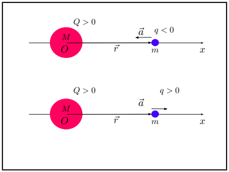
and

$$m = m_1 + m_2$$

Is this a fundamental Law? For this to be a 'Law', it must hold under any situation, not just for springs!

Electrostatic Force

We take objects of different mass (measured through springs). We have a 'reference' object of mass M much larger than any other mass, kept at rest. We 'spray' some charge Q on this reference object and different charges on the other objects.



Observations

- Acceleration of object of mass m depends upon its position relative to the reference object. It is directed opposite to displacement m relative to M and its magnitude varies as inverse square of the distance For smadisplacements, it is found that

$$\vec{a} \propto -\frac{\vec{r}}{r^3}$$

- For the same charge, acceleration is inversely proportional to mass m.
- For the same mass, acceleration is proportional to each charge (Q and q)

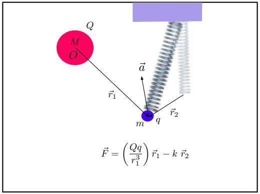
Once again we can summarise these observations in the form of an equation

$$\vec{F} = m\,\vec{a}$$

where

$$\vec{F} = \left(\frac{Qq}{r^3}\right)\vec{r}$$

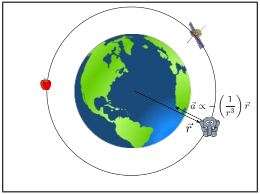
More combinations, same Law



$$\begin{aligned}\vec{F} &= \vec{F}_1 + \vec{F}_2 \\ &= \left(\frac{Qq}{r_1^3}\right)\vec{r}_1 - k\,\vec{r}_2\end{aligned}$$

Gravitational Force

Gravitational 'force'



Earth attracts all objects towards its centre with a 'force' such that

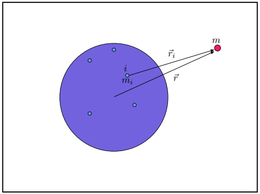
$$\vec{a} \propto -\left(\frac{1}{r^3}\right)\vec{r}$$

independent of the mass of the objects!
If we imagine shrinking the Earth down to a point, this force should still be there, though perhaps its form could in principle be different. However, it would still be directed towards the centre of the Earth.

Visualise Earth as an aggregate of tiny (point-like) objects, each exerting some gravitational force on a test point-like object of (inertial) mass m. This force should be of the form

$$\vec{F}_i = -\alpha\, f(r_i)\, \vec{r}_i$$

where $\vec{r}_i$ is the displacement of the test object from the ith constituent and $f(r_i)$ is some function of r_i. The constant α should be the same for all the constituents, since they are assumed to be identical



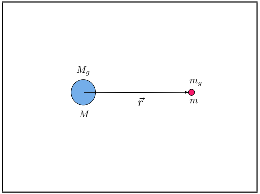
Assuming the total force due to the Earth is a vector sum of such forces, we should have

$$\sum_i f(r_i)\, \vec{r}_i \propto \left(\frac{1}{r^3}\right) \vec{r}$$

The only function $f(r)$ which will give this is

$$f(r) \propto \frac{1}{r^3}$$

whose magnitude also goes as inverse distance squared! Since the force is so similar to the electrostatic force (but universal), we conjecture that all point-like objects possess a ‘gravitational charge’ m_g such that the force exerted by an object of charge M_g on an object with charge m_g is



$$\vec{F}_m = \left(\frac{GM_g m_g}{r^3}\right)\vec{r}$$

If we fix object with (inertial mass) M and test this relation for different test masses m, the acceleration of such a test object will be

$$\begin{aligned} \vec{a} &= \frac{\vec{F}}{m} \\ &= \left(\frac{GM_g}{r^3}\right)\left(\frac{m_g}{m}\right)\vec{r} \end{aligned}$$

Experimentally, this acceleration is found to be independent of what is used as a test object! This is only possible if

$$m = m_g$$

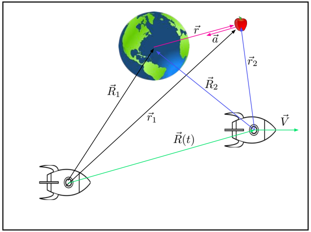
Thus, gravitational charge is the same as inertial mass. This is also known as the *Principle of Equivalence* and is the starting point of Einstein's *General Theory of Relativity*.

Galilean Relativity

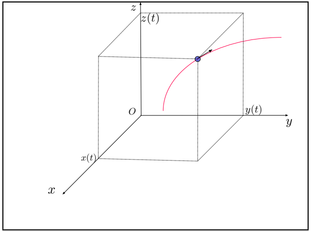
Galilean Relativity: The same experiment conducted in different inertial frames will give the same result. This implies that there is no way to experimentally single out any one inertial frame from another. This will happen only if the *Laws of Physics have the same form in all inertial frames*.
Does the Second Law have the same form in all Inertial frames?

$$\vec{F} = m\,\vec{a}$$

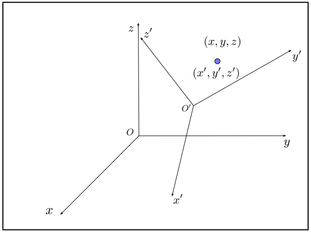
Since $\vec{a}$ is measured the same in all inertial frames (in uniform relative motion), force $\vec{F}$ must be a vector which remains unchanged from one inertial frame to another. Since all the forces considered so far are proportional to displacement, this is true (displacement vector is measured to be the same by all observers)



Motion in three dimensions

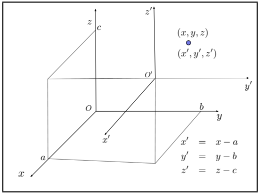


Different Cartesian coordinate systems differing by translation and rotation

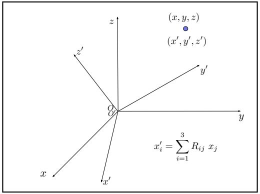


Translations and Rotations

Translations:



Rotations:



$(x_1, x_2, x_3) \equiv (x, y, z)$ etc.

Rotation about z:

$$\begin{aligned} x_1' &= x_1 \cos\theta + x_2 \sin\theta \\ x_2' &= -x_1 \sin\theta + x_2 \cos\theta \\ x_3' &= x_3 \end{aligned}$$

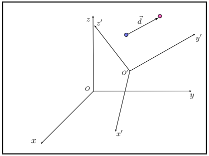
Rotation Matrices:

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Example: Rotation about x,y,z

$$\begin{aligned} R_x(\theta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \\ R_y(\theta) &= \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \\ R_z(\theta) &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Transformation of Displacement Components:



$$\Delta x_i' = \sum_{i=1}^{3} R_{ij}\, \Delta x_j$$

We can write this as

$$\Delta X' = R\, \Delta X$$

where

$$\Delta X = \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{pmatrix}$$

and

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$$

Orthogonal Property of Rotation Matrices: Since rotations and translations preserve length of displacement, it follows that

$$\Delta x_1'^2 + \Delta x_2'^2 + \Delta x_3'^2 = \Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2$$

This can be written as

$$\Delta X'^T \Delta X' = \Delta X^T \Delta X$$

where ΔX^T is the transpose of ΔX. It then follows that

$$\begin{aligned} \Delta X'^T \Delta X' &= \Delta X^T R^T R \Delta X \\ &= \Delta X^T \Delta X \end{aligned}$$

which imples

Orthogonal Property of Rotation Matrix

$$R^T R = I$$

where I is the Identity matrix.

Definition

A 'Vector' $\vec{A}$ (in a plane) is any physical quantity that in any given Cartesian coordinate system (x, y) is represented by a three numbers (A_x, A_y, A_z) such that these numbers transform between coordinate systems as

$$A'_i = \sum_{i=1}^{3} R_{ij}\, A_j$$

Definition

Scalar A scalar is any quantity that is measured to be the same in all coordinate systems differing by a translation or rotation.

Scalar Product: A scalar out of two vectors Define

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

We define column vectors

$$\begin{aligned} A &= \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \\ B &= \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} A'_x B'_x + A'_y B'_y + A'_z B'_z &= A'^T B' \\ &= A^T R^T R B \\ &= A^T B \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned}$$

which demonstrates that $\vec{A} \cdot \vec{B}$ is a scalar.

Cross Product

In three dimensions, given two vectors $\vec{A}$ and $\vec{B}$, we can construct another vector $\vec{C}$ through the so-called *cross-product*

$$\vec{C} = \vec{A} \times \vec{B}$$

such that in a given coordinate system, if the components of $\vec{A}$ are (A_x, A_y, A_z) and the components of $\vec{B}$ are (B_x, B_y, B_z), the the components of $\vec{C}$ are

$$\begin{aligned} C_x &= A_y B_z - A_z B_y \\ C_y &= A_z B_x - A_x B_z \\ C_z &= A_x B_y - A_y B_x \end{aligned}$$

How do we know the set of numbers (C_x, C_y, C_z) describe a vector? It is easy to check that under rotations, these numbers transform the same way as a vector should. That is, given that $A'_i = \sum_{i=1}^{3} R_{ij}\ A_j$ and $B'_i = \sum_{i=1}^{3} R_{ij}\ B_j$ it follows that

$$C'_i = \sum_{i=1}^{3} R_{ij}\ C_j$$

Let us check this for rotation about z-axis. We are given that

$$\begin{aligned} A'_x &= A_x \cos\theta + A_y \sin\theta \\ A'_y &= -A_x \sin\theta + A_y \cos\theta \\ A'_z &= A_z \end{aligned}$$

and that

$$\begin{aligned} B'_x &= B_x \cos\theta + B_y \sin\theta \\ B'_y &= -B_x \sin\theta + B_y \cos\theta \\ B'_z &= B_z \end{aligned}$$

Then,

$$\begin{aligned} C'_x &= A'_y B'_z - A'_z B'_y \\ &= (-A_x \sin\theta + A_y \cos\theta) B_z - (-B_x \sin\theta + B_y \cos\theta) A_z \\ &= (A_y B_z - A_z B_y) \cos\theta + (A_z B_x - A_x B_z) \sin\theta \\ &= C_x \cos\theta + C_y \sin\theta \end{aligned}$$

Similarly,

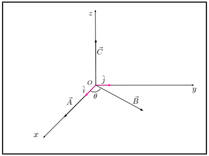
$$
\begin{aligned}
C'_y &= A'_z B'_x - A'_x B'_z \\
&= (B_x \cos\theta + B_y \sin\theta) A_z - (A_x \cos\theta + A_y \sin\theta) B_z \\
&= -(A_y B_z - A_z B_y) \sin\theta + (A_z B_x - A_x B_z) \cos\theta \\
&= -C_x \sin\theta + C_y \cos\theta
\end{aligned}
$$

and

$$
\begin{aligned}
C'_z &= A'_x B'_y - A'_y B'_x \\
&= (A_x \cos\theta + A_y \sin\theta)(-B_x \sin\theta + B_y \cos\theta) \\
&- (-A_x \sin\theta + A_y \cos\theta)(B_x \cos\theta + B_y \sin\theta) \\
&= A_x B_y - A_y B_x \\
&= C_z
\end{aligned}
$$

which verifies that (C_x, c_y, C_z) transform as components of a vector under rotation about the z direction. It is easily checked that the same is true for rotations about the x and y directions also.

To visualise the cross product of vectors $\vec{A}$ and $\vec{B}$, let us choose a coordinate system such that the two vectors lie in the $x-y$ plane, with $\vec{A}$ along the x direction. Then, $\vec{A} = \left|\vec{A}\right| \ \hat{i}$ and $\vec{B} = \left|\vec{B}\right| \cos\theta\,\hat{i} + \left|\vec{B}\right| \sin\theta\,\hat{j}$ where θ is the angle between $\vec{A}$ and $\vec{B}$

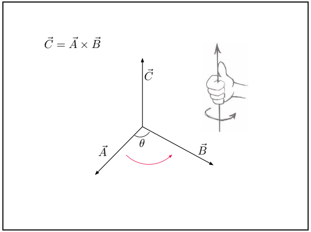


In this coordinate system, $C_x = C_y = 0$, and $C_z = \left|\vec{A}\right| \ \left|\vec{B}\right| \sin\theta$. This tells us that $\vec{C} = \vec{A} \times \vec{B}$ is perpendicular to the plane containing both $\vec{A}$ and $\vec{B}$ and its magnitude is given by the product of the magnitudes of the two vectors times sin of the angle between them. Since this description is coordinate independent, this is the interpretation of the cross -product of two vectors

Definition

Cross Product The cross product of two vectors $\vec{A}$ and $\vec{B}$ is a vector perpendicular to both $\vec{A}$ and $\vec{B}$ and has magnitude equal to the product of the magnitudes of the two vectors times sin of the angle between them.

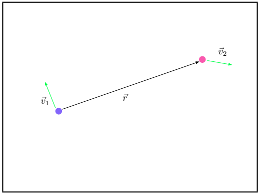
Note that this leaves an ambiguity so far as the direction of the resulting vector is concerned. However, given the definition of the cross product in terms of components, this definition implies the 'right-hand screw rule'



Magnetic Force

It is observed that two current carrying wires attract each other. In particular, if one of them is held rigidly, a charged particle in its vicinity will accelerate, *if it has a non-zero velocity*. these observations are consistent with the following hypothesis: Moving charged particles exert a force in addition to the electrostatic force, which is proportional to their velocities (and charges). A particle with charge q_1 and moving with velocity $\vec{v}_1$ exerts a force on another charged particle with charge q_2 moving with velocity $\vec{v}_2$ which is proportional to their velocities and falls off inversely with square of the distance between the charges. Clearly, this force has to be a vector quantity, constructed out of vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{r}$ where $\vec{r}$ is the displacement from one charge to another. The expression for this magnetic force is

$$\vec{F}_m = \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^3}\,\vec{v}_2 \times (\vec{v}_1 \times \vec{r})$$



Lorentz Force

The total force exerted by a charge q_1 on q_2 is then

$$\begin{aligned}\vec{F} &= \frac{q_1 q_2}{4\pi\epsilon_0 r^3}\vec{r} + \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^3}\,\vec{v}_2 \times (\vec{v}_1 \times \vec{r}) \\ &= q_2\left(\vec{E} + \vec{v}_2 \times \vec{B}\right)\end{aligned}$$

where we have defined auxilliary quantities $\vec{E}$ and $\vec{B}$ (electric and magnetic fields produced by charge q_1 at the location of charge q_2)

$$\begin{aligned}\vec{E} &= \frac{q_1}{4\pi\epsilon_0 r^3}\vec{r} \\ \vec{B} &= \frac{\mu_0}{4\pi}\frac{q_1}{r^3}\,(\vec{v}_1 \times \vec{r})\end{aligned}$$

This is the so-called *Lorentz Force*. Note that as introduced here, there is nothing 'real' about $\vec{E}$ and $\vec{B}$, they are just artificial constructions.

The convenience of introducing the concept of electric and magnetic fields is that if there are charges $q_i; i = 1, 2.., N$ moving with velocities $\vec{v}_i; i = 1, 2, ..N$ in the vicinity of a charge q moving with velocity $\vec{v}$, the total force exerted by them (using the Principle of Superposition of Forces we had discovered) on charge q can still be written down this way

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

where

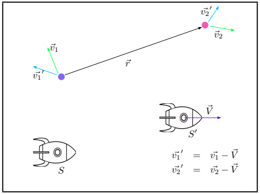
$$\begin{aligned} \vec{E} &= \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon_0 r_i^3}\vec{r}_i \\ \vec{B} &= \sum_{i=1}^{N} \frac{\mu_0}{4\pi}\frac{q_i}{r_i^3}\left(\vec{v}_i \times \vec{r}_i\right) \end{aligned}$$

and $\vec{r}_i$ is the displacement vector from q_i to q.

End of Newtonian Physics

It is easy to see that the Lorentz Force is not measured to be the same in all inertial frames. Consider once again the force exerted by charge q_1 on charge q_2, as measured in some inertial frame S

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^3}\vec{r} + \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^3}\ \vec{v_2} \times \left(\vec{v_1} \times \vec{r}\right)$$



In frame S', the force will be measured to be

$$\vec{F}' = \frac{q_1 q_2}{4\pi\epsilon_0 r^3}\vec{r} + \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^3}\ \vec{v_2}' \times \left(\vec{v_1}' \times \vec{r}\right)$$

It is easy to see that $\vec{F}' \neq \vec{F}$. For definiteness, if $\vec{v_1} = \vec{v_2} = 0$, then $\vec{v_1}' = \vec{v_2}' = -\vec{V}$, so that

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^3}\vec{r}$$

but

$$\vec{F}' = \frac{q_1 q_2}{4\pi\epsilon_0 r^3}\vec{r} + \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^3}\;\vec{V} \times \left(\vec{V} \times \vec{r}\right)$$

in which the magnetic term need not be zero. Then, the presence of a magnetic force destrys Galilean Invariance of the interaction between thwo charged particles.
To discover what has gone wrong, let us compute the relative magnitudes of the electrostatic and magnetic forces between two charged particles. Then

$$\frac{\left|\vec{F}_M\right|}{\left|\vec{F}_E\right|} = \mu_0\epsilon_0 \left|\vec{v_2} \times \left(\vec{v_1} \times \hat{r}\right)\right|$$

If we numerically compute the product $\mu_0\epsilon_0$, it turns out to be given by

$$\mu_0\epsilon_0 = \frac{1}{c^2}$$

where c is the speed of light! The ratio of the two forces is then of the order of v^2/c^2 where v is the order of velocity of the charges. This is negigible, unless $v \sim c$.

The sudden appearance of c in our theory (and the fact that our theory seems to be inconsistent with a magnetic force) tells us that as long as we are dealing with speeds which are much less than the speed of light (or, equivalently, in the limit $c \to \infty$), Newton's Laws will be valid. Else, our theory is inconsistent with the Principle of Relativity (Laws of Physics should appear the same in all Inertial Frames). What has gone wrong if c is finite? Experimentally, the magnetic force is observed, so we cannot ignore it. What needs to change is our view of transformations of position, velocity and acceleration from one inertial frame to another. Experimentally, it is observed that *the speed of light is measured to be the same by observers*, which is in contradiction with the obvious fact that observers in relative motion will observe velocity of objects to be different.

The resolution to this problem lies in a complete overhaul of our understanding of space and time. This is the framework of the Theory of Relativity, which we will visit later. For now, the following should be sufficient:

Since the speed of light is finite and observed to be the same by all observers, a consequence of this 'speed limit' is that no information can travel faster than light. This completely rules out the idea that objects exert forces on each other. For, if the force between two charged particles at rest is given by the Coulomb Force (which depends on the instantaneous displacement of the two charges), if we suddenly change the position of one charge, since the displacement to the other charge changes, the force will change instantaneously, changing the acceleration of the other charge, in effect transmitting this influence instantaneously!

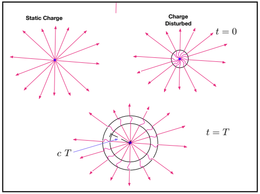
Fields

If we cannot say any longer that objects exert forces on other objects, how are we to explain the origin of acceleration, which somehow *does* depend on interaction with other objects? The resolution to this problem lies in a change of point of view (and the resulting dynamics): Charges particles interact not with each other, but Electric and Magnetic fields, which cannot be treated to be artificial entities, but have to be thought of as 'real' dynamical systems in themselves. The force experienced by a charged particles is *still* given by the Lorentz force law

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields *at the location of the charge*. However, these fields are not 'produced' by the other charge, but are independent dynamical entities which satisfy dynamical equations of their own (Maxwell Equations of electrodynamics).

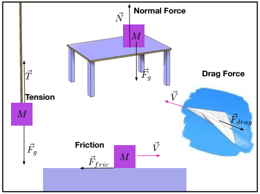
The most important consequence of these equations is that if we disturb the first charge, it disturbs the fields in its vicinity. This disturbance travels as a ripple in the lectric and magnetic fields which fill space at the *speed of light*. when this ripple reaches the other charge, it changes the force on it (and therefore its acceleration)



This is completely consistent with an extremely central principle in Physics: the *Principle of Locality*. This principle states that the motion of an object can be influenced only by other 'objects' in its immediate neighbourhood. In case of charged objects, these other 'objects' happen to be electric and magnetic fields in their immediate vicinity.

Other Forces

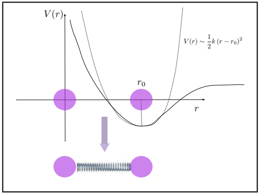
What about (a host of other) forces?



These are all ***effective*** macroscopic forces arising out of the **Electromagnetic Force** between atoms (whose description needs to be modified according to the Laws of **Quantum Mechanics**). Do we then need to know Quantum Mechanics to describe these forces? Not if we are only interested in their effect at a macroscopic scale, not at the scale of atoms (or smaller).

Short Distance Forces

Electromagnetic force is a 'long-distance' force, since it does not have a range. However, since at the level of atoms, Quantum effects become important, Planck's constant induces length scales which lead to a finite range interaction between neutral atoms/molecules

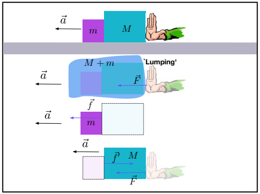


In solids, we can often pretend that constituent atoms/molecules are point-like objects connected by springs. This approximately accounts for elasticity of solids (though not deformation or thermal expansion).

Effective Macroscopic Forces

- **Tension in strings/ropes**: Can be visualised as springs with a very large spring constant (so that a small amount of stretching can result in an appreciable force).
- **Normal Force:** Solid surfaes can be visualised as spring mattresses which are deformed slightly when an object is pressed against them. This deformation gives rise to a force perpendicular to the surface.
- **Friction:** Macroscopic force arisng out of microscopic collisions with atoms/molecules of the spring mattress. Energy is lost to these collisions which results in the mattress vibrating, the vibrations thermalising into 'heat energy'.
- **Drag Forces:** Arise due to interaction with atoms/molecules of a fluid medium. So long as the velocity of the object moving through the fluid is much smaller than the velocity of sound in the medium, this force approximately has magnitude proportional to the velocity of the object.

An Illustration of 'Lumping' masses



Applying Second Law to the two objects 'lumped' together

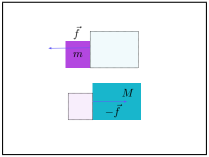
$$\vec{F} = (M + m)\,\vec{a}$$

Applying Second Law to mass m

$$\vec{f} = m\,\vec{a}$$

Applying Second Law to mass M

$$\begin{aligned} \vec{F} + \vec{f}' &= M\,\vec{a} \\ \Longrightarrow \vec{f}' &= M\,\vec{a} - \vec{F} \\ &= M\,\vec{a} - (M+m)\,\vec{a} \\ &= -m\vec{a} \end{aligned}$$



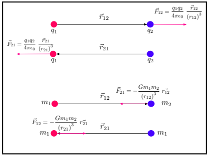
We then discover that *'Force exerted by m on M is equal in magnitude but opposite in direction'*. This can easily be seen to be valid in general situations when objects are in contact with each other (exerting *Contact Forces*).
Have we discovered a LAw which is not fundamental but a consequence of the first two laws?
No. We are *not* allowed to apply the first two Laws to extended objects. We can apply them to only point-like objects, since only these can possess attributes such as 'position', 'velocity' and 'acceleration'. When we try to extend them to objects with size (and also 'lump' them together to create even bigger objects), we are inadvertently *assuming* the validity of a *third* Law:

Third Law

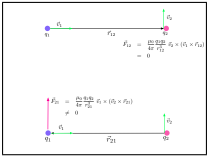
Particles exert forces that are equal in magnitude but opposite in direction on each other.

Fundamental Forces and the Third Law

The Electrostatic force and the Gravitational force are clearly consistent with the Third Law



Magnetic force violates the Third Law!



Momentum

What good is then the third 'Law'?
If we rewrite the Second Law, we can discover an alternative to the Third Law, which *is* believed to be a fundamental Law of Nature. the (total) force acting on a particle of mass m can be rewritten as

$$\begin{aligned} \vec{F} &= m\vec{a} \\ &= m\frac{d\vec{v}}{dt} \\ &= \frac{d\left(m\,\vec{v}\right)}{dt} \\ &= \frac{d\vec{p}}{dt} \end{aligned}$$

where we define $\vec{p} = m\,\vec{v}$ as *momentum* of the particle. The advantage of defining this physical quantity is that if we assume that the Third Law is correct, it tells us that given two or more particles, if we define the *total momentum* of the system as the vector sum of momentum of all the particles, then this quantity is *conserved*, that is, does not change with time, even though the overall motion of all the particles may be complicated.

Conservation of Momentum

Assume there are particles such that the force exerted by the i^{th} particle on the j^{th} particle is $\vec{F}_{ij}$. Define the total momentum of the system as

$$\vec{P} = \sum_i \vec{p}_i$$

Then

$$\begin{aligned} \frac{d\vec{P}}{dt} &= \sum_i \frac{d\vec{p}_i}{dt} \\ &= \sum_i \sum_{j \neq i} \vec{F}_{ji} \end{aligned}$$

If the Third Law is correct, this is zero, with force terms cancelling in pairs. For instance, for three particles, we get

$$\begin{aligned} \frac{d\vec{P}}{dt} &= \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{21} + \vec{F}_{23} + \vec{F}_{31} + \vec{F}_{32} \\ &= \left(\vec{F}_{12} + \vec{F}_{21}\right) + \left(\vec{F}_{13} + \vec{F}_{31}\right) + \left(\vec{F}_{23} + \vec{F}_{32}\right) \\ &= 0 \end{aligned}$$

Third Law (Again)

We can then restate the Third Law as follows

Third Law

Given a system of interacting particles, the total momentum of the system is conserved.

For a system of charged particles, clearly the total momentum of the system is not conserved, since there are magnetic forces which do not cancel in action-reaction pairs. What then is the point of rephrasing the Third Law in terms of momentum? What is the point of defining momentum at all?

Fields Carry Momentum

Electric and Magnetic fields carry momentum. At any instant, the total momentum of charged particles **plus** the fields is conserved.

We need to treat the Thid Law of Motion as an approximation which is a good one so long as the speed of particles is much smaller than the speed of light and the size of objects is such that the time it would take light to travel across them is much smaller than the duration of the experiment (which is usually the case). In all applications we are interested in, this will be true. Then, the Third Law of Motion (*"To every action there is an equal and opposite reaction"*) and conservation of momentum are equivalent in such situations. Note that the conservation of momentum is believed to be a true Law of Nature, it is just that in such situations it will agree with the Third Law (which is not strictly a Law of Nature).

Classical Mechanics

A. Gupta

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St. Stephen's College

Outline

Newton's Second Law as Differential Equations

Consider a particle of mass m moving under the influence of a force which depends on the position and velocity of the particle. We can cast the Second Law ($\vec{F} = d\vec{p}/dt$) as coupled differential equations

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{1}{m}\,\vec{p} \\ \frac{d\vec{p}}{dt} &= \vec{F}\,(\vec{r},\vec{p}) \end{aligned}$$

Given position $\vec{r}$ and momentum $\vec{p}$ at some instant t_0, we can use these relations to calculate them at any other instant through the following recursive algorithm

$$\begin{aligned} \vec{r}(t+\Delta t) &= \vec{r}(t) + \frac{1}{m}\,\vec{p}(t)\,\Delta t \\ \vec{p}(t+\Delta t) &= \vec{p}(t) + \vec{F}\left(r(\vec{t}), p(\vec{t})\right)\,\Delta t \end{aligned}$$

To use this algorithm on a computer, we set up a Cartesian coordinate system. In this coordinate system, these reduce to three pairs of coupled differential equations, one pair for each coordinate direction

$$
\begin{aligned}
x(t+\Delta t) &= x(t) + \frac{1}{m}\, p_x(t)\, \Delta t \\
p_x(t+\Delta t) &= p_x(t) + F_x\left[x(t), y(t), z(t), p_x(t), p_y(t), p_z(t)\right]\ \Delta t \\
y(t+\Delta t) &= y(t) + \frac{1}{m}\, p_y(t)\, \Delta t \\
p_y(t+\Delta t) &= p_y(t) + F_y\left[x(t), y(t), z(t), p_x(t), p_y(t), p_z(t)\right]\ \Delta t \\
z(t+\Delta t) &= z(t) + \frac{1}{m}\, p_z(t)\, \Delta t \\
p_z(t+\Delta t) &= p_z(t) + F_z\left[x(t), y(t), z(t), p_x(t), p_y(t), p_z(t)\right]\ \Delta t
\end{aligned}
$$

Given $x(t_0), y(t_0), z(t_0), p_x(t_0), p_y(t_0), p_z(t_0)$, we can use these recursively to calculate $x(t), y(t), z(t), p_x(t), p_y(t), p_z(t)$ for any time t by dividing the interval $T = t - t_0$ into N 'small' intervals $\Delta t = T/N$. The smaller Δt, the more accurate the result (of course, more the number of calculations to be performed).

Illustration: 1-D Harmonic Oscillator

1-D Harmonic Oscillator: Particle attracted towards a point with force proportional to displacement from the point and directed towards it The coupled equations are

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{m}\,p \\ \frac{dp}{dt} &= -k\,x\end{aligned}$$

Assume the following initial conditions: at $t = 0, x_0 = A, p_0 = 0$. There is a natural time scale in the problem: $T_0 = \sqrt{m/k}$. There is also a natural length scale, the initial distance from the point of attraction, namely A. We measure time measured in units of this fundamental time scale as τ and position measured in units of this length scale as $\tilde{x}$. Then, $\tilde{x} = x/A$ and $\tau = t/T_0$. Substituting these in the coupled equations, we get

$$\frac{d\tilde{x}}{d\tau} = \frac{1}{A\sqrt{mk}}\, p$$
$$\frac{1}{A\sqrt{mk}}\frac{dp}{d\tau} = -\tilde{x}$$

It is easy to see that the combination $P = A\sqrt{mk}$ has dimension of momentum. then, we can measure momentum in units of P, such that $\tilde{p} = p/P$. Then, we get the following coupled equations

$$\frac{d\tilde{x}}{d\tau} = \tilde{p}$$
$$\frac{d\tilde{p}}{d\tau} = -\tilde{x}$$

which we need to solve, given $\tilde{x}(0) = 1$ and $\tilde{p}(0) = 0$.

A Conserved quantity

Note that the equations of motion predict that even though position and momentum change with time, there is a certain function of position and momentum which does *not* change with time, i.e., is ***Conserved***

$$H(\tilde{x}, \tilde{p}) = \frac{1}{2}\tilde{p}^2 + \frac{1}{2}\tilde{x}^2$$

To see that it is conserved, we compute its derivative with respect to time

$$\begin{aligned} \frac{dH}{d\tau} &= \frac{1}{2}\frac{d\tilde{p}^2}{d\tau} + \frac{1}{2}\frac{d\tilde{x}^2}{d\tau} \\ &= \tilde{p}\frac{d\tilde{p}}{d\tau} + \tilde{x}\frac{d\tilde{x}}{d\tau} \\ &= \tilde{p}(-\tilde{x}) + \tilde{x}(\tilde{p}) \\ &= 0 \end{aligned}$$

where we have used the equations of motion. The funciton H is proportional to the total mechanical energy of the system.

Naive Algoritm

Naively, we would use the following algorithm

$$\begin{aligned}\tilde{x}(\tau+\Delta\tau) &= \tilde{x}(\tau)+\tilde{p}(\tau)\Delta\tau \\ \tilde{p}(\tau+\Delta\tau) &= \tilde{p}(\tau)-\tilde{x}(\tau)\Delta\tau\end{aligned}$$

with the starting condition $\tilde{x}(0)=1$ and $\tilde{p}(0)=0$. However, since $\Delta\tau$ can never be 'infinitesimal', this procedure will introduce an error of the order of $\Delta\tau^2$ at every step. Then, if the algorithm is carried out upto N steps (equivalently, upto time $\tau=N\Delta\tau$), the total error will be of the order $N\Delta\tau^2=\tau\ \Delta\tau$ which will increase liearly with τ. Then, the longer the duration for which the motion is considered, the larger the error. Let us write a Python program for this algorithm and plot the position and momentum as functions of time.

Python Program for Naive Algorithm

```
from pylab import * ## Imports a plotting program.


########## Initialisation ############################################
position = 1.0    ## Initial position
momentum = 0.0    ## Initial momentum
time = 0.0
end_time = 20.0   ## Time upto which position and momentum are to be
N = 200           ## Number of time divisions
delta_t = end_time/N     ## Time increment
position_list = [position]  ## Creates a list of positions and popul
momentum_list = [momentum]  ## Creates a list of momenta and populat
time_list = [time]          ## Creates a list of time instants and p

##########    Iterations in position and momentum ##################
for i in range(0,N):
        new_position = position + (momentum*delta_t) # Updates posit
        new_momentum = momentum - (position*delta_t) # Update moment
        position = new_position
        momentum = new_momentum
        time = time + delta_t
        position_list.append(position)
        momentum_list.append(momentum)
        time_list.append(time)

#### Plotting the lists ############################################
xlabel('t')
ylabel('x,p)')
title('Naive Harmonic Oscillator')
plot(time_list,position_list,marker = 'o', label = 'x')
plot(time_list,momentum_list,marker = 's',label = 'p')
legend()
show()
```

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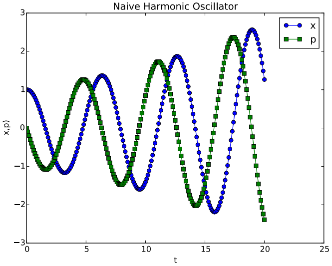

A good check on the accuracy of the algorithm will be to check if it keeps the energy function H constant

```
from pylab import * ## Imports a plotting program.

########## Initialisation ##########################################
position = 1.0    ## Initial position
momentum = 0.0    ## Initial momentum
energy = (position*position)/2 + (momentum*momentum)/2 ## Initial en
time = 0.0
end_time = 20.0    ## Time upto which position and momentum are to be
N = 200            ## Number of time divisions
delta_t = end_time/N     ## Time increment
energy_list = [energy]
time_list = [time]          ## Creates a list of time instants and p

##########    Iterations in position and momentum #################
for i in range(0,N):
        new_position = position + (momentum*delta_t) # Updates posit
        new_momentum = momentum - (position*delta_t) # Update moment
        position = new_position
        momentum = new_momentum
        energy = (position*position)/2 + (momentum*momentum)/2
        time = time + delta_t
        energy_list.append(energy)
        time_list.append(time)

#### Plotting the lists ###########################################
xlabel('t')
ylabel('Energy')
title('Naive Harmonic Oscillator - Energy')
plot(time_list,energy_list,marker = 'o')
show()
```

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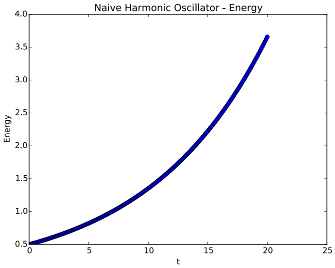


Problem with Naive Algorithm

Taylor Series:

$$f(x_0 + a) = f(x_0) + a\, f'(x_0) + \frac{a^2}{2!}\, f''(x_0) + \frac{a^3}{3!}\, f'''(x_0) + \ldots.$$

Then

$$\begin{aligned} x(t+\Delta t) &= x(t) + \Delta t\, \dot{x}(t) + \frac{\Delta t^2}{2}\, \ddot{x}(t) + \ldots \\ p(t+\Delta t) &= p(t) + \Delta t\, \dot{p}(t) + \frac{\Delta t^2}{2}\, \ddot{p}(t) + \ldots \end{aligned}$$

In the naive algorithm, we ignore terms $\mathcal{O}(\Delta t^2)$ and higher in expansions for position and momentum, such that

$$\begin{aligned} x(t+\Delta t) &= x(t) + \Delta t\, \dot{x}(t) + \mathcal{O}(\Delta t^2) \\ p(t+\Delta t) &= p(t) + \Delta t\, \dot{p}(t) + \mathcal{O}(\Delta t^2) \end{aligned}$$

This is why we accumulate an error of $\mathcal{O}(\Delta t^2)$ in every step.

Leapfrog Algorithm

The 'Leapfrog' Algorithm:
This algorithm is based on the observation

$$f(t+\Delta t) = f(t) + \Delta t\, \dot{f}\,(t+\Delta t/2) + \mathcal{O}(\Delta t^3)$$

This follows since

$$\dot{f}\,(t+\Delta t/2) = \dot{f}(t) + \frac{\Delta t}{2}\,\ddot{f}(t) + \mathcal{O}(\Delta t^2)$$

so that

$$\begin{aligned}
f(t) + \Delta t\, \dot{f}\,(t+\Delta t/2) + \mathcal{O}(\Delta t^3) &= f(t) + \Delta t\, \left\{ \dot{f}(t) + \frac{\Delta t}{2}\,\ddot{f}(t) + \mathcal{O}(\Delta t^2) \right\} + \mathcal{O}(\Delta t^3) \\
&= f(t) + \Delta t\, \dot{f}(t) + \frac{\Delta t^2}{2}\,\ddot{f}(t) + \mathcal{O}(\Delta t^3) \\
&= f(t+\Delta t) + \mathcal{O}(\Delta t^3)
\end{aligned}$$

We exploit this algorithm as follows for the Harmonic Oscillator: We divide the total time interval into N slices of width Δt as before. Then, using the new approximation, the position at instant $t + \Delta t$ is given as

$$\begin{aligned}\tilde{x}(\tau + \Delta\tau) &= \tilde{x}(\tau) + \Delta\tau\ \dot{\tilde{x}}(\tau + \Delta\tau/2) \\ &= \tilde{x}(\tau) + \Delta\tau\ \tilde{p}(\tau + \Delta\tau/2)\end{aligned}$$

Then, the position at $\tau + \Delta\tau$ depends on momentum at $\tau + \Delta\tau/2$. Say, we know the momentum at $\tau + \Delta\tau/2$. Then, we can calculate it at $\tau + 3\Delta\tau/2$ using the same approximation

$$\begin{aligned}\tilde{p}(\tau + 3\Delta\tau/2) &= \tilde{p}(\tau + \Delta\tau/2) + \Delta\tau\ \dot{\tilde{p}}(\tau + \Delta\tau) \\ &= \tilde{p}(\tau + \Delta\tau/2) - \Delta\tau\ \tilde{x}(\tau + \Delta\tau)\end{aligned}$$

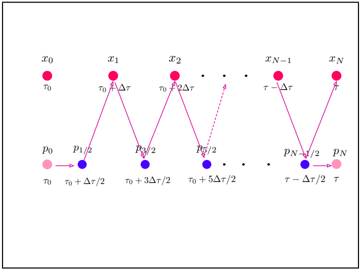
This now allows us to calculate the position at $\tau + 2\Delta\tau$ which then allows us to calculate the momentum at $\tau + 5\Delta\tau/2$ and so on. At the start of the algorithm, we are given $\tilde{x}(\tau_0)$ and $\tilde{p}(\tau_0)$. We first 'seed' the momentum at $\tau = \tau_0 + \Delta\tau/2$ using the original approximation

$$\begin{aligned}\tilde{p}(\tau_0 + \Delta\tau/2) &= \tilde{p}(\tau_0) + \Delta\tau/2\ \dot{\tilde{p}}(\tau_0) + \mathcal{O}(\Delta\tau^2) \\ &= \tilde{p}(\tau_0) - \Delta\tau/2\ \tilde{x}(\tau_0) + \mathcal{O}(\Delta\tau^2)\end{aligned}$$

Note that even though we have introduced an error here of $\mathcal{O}(\Delta\tau^2)$, this is introduced only **once** and does not accumulate as before. We can now evaluate (using the above algorithm recursively) the position at the final instant τ and the momentum at instant $\tau - \Delta\tau/2$. We can once more use the original approximation to 'push' this momentum upto instant τ

$$\begin{aligned} \tilde{p}(\tau) &= \tilde{p}(\tau - \Delta\tau/2) + \Delta\tau/2\, \dot{\tilde{p}}(\tau - \Delta\tau) + \mathcal{O}(\Delta\tau^2) \\ &= \tilde{p}(\tau - \Delta\tau/2) - \Delta\tau/2\, \tilde{x}(\tau - \Delta\tau) + \mathcal{O}(\Delta\tau^2) \end{aligned}$$

which once again introduces error of order $\mathcal{O}(\Delta\tau^2)$. Then, we have introduced this error only twice.



Improved Harmonic Oscillator Program

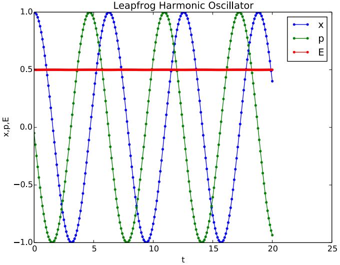
```
from pylab import * ## Imports a plotting program.


########## Initialisation ########################################
position = 1.0    ## Initial position
initial_momentum = 0.0    ## Initial momentum
time = 0.0
end_time = 20.0   ## Time upto which position and momentum are to
N = 200           ## Number of time divisions
delta_t = end_time/N     ## Time increment
momentum = initial_momentum - (delta_t*position)/2 ## `Seeded' mor
energy = (position*position)/2 + (momentum*momentum)/2 ## Initial
position_list = [position]  ## Creates a list of positions and pop
momentum_list = [momentum]  ## Creates a list of momenta and popu
energy_list = [energy]
time_list = [time]          ## Creates a list of time instants anc

##########    Iterations in position and momentum ###############
for i in range(0,N):
        new_position = position + (momentum*delta_t) # Updates pos
        new_momentum = momentum - (new_position*delta_t) # Updates
        position = new_position
        momentum = new_momentum  ## This is not at the same instai
        temp_position = position + (delta_t*momentum)/2 ## This pi
        energy = (temp_position*temp_position)/2 + (momentum*momer
        time = time + delta_t
        position_list.append(position)
        momentum_list.append(momentum)
        energy_list.append(energy)
        time_list.append(time)

#### Plotting the lists ##########################################
xlabel('t')
ylabel('x,p,E')
title('Leapfrog Harmonic Oscillator')
plot(time_list,position_list,marker = '.', label = 'x')
plot(time_list,momentum_list,marker = '.',label = 'p')
plot(time_list,energy_list,marker = '.',label = 'E')
legend()
show()
```

1



Planetary Motion

Consider a massive spherical object (such as the Sun) with mass M with its centre at the origin, and an object of mass m moving under its gravitational influence. We assume that $M >> m$ and assume that the massive object is at rest. The equation of motion of the object of mass m will be

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{GMm}{r^3}\,\vec{r}$$

where $\vec{r}$ is the position vector of the object relative to the massive object. The mass m of the object cancels on both sides of the equation, so that we have

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{r^3}\,\vec{r}$$

We write this as coupled equations

$$\begin{aligned}\frac{d\vec{r}}{dt} &= \vec{v}\\ \frac{d\vec{v}}{dt} &= -\frac{GM}{r^3}\,\vec{r}\end{aligned}$$

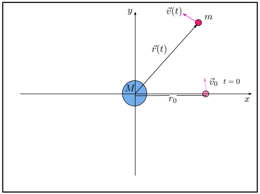
There are two conserved quantities, proportional to the energy and angular momentum of the object

$$E = \frac{1}{2}\vec{v}^2 - \frac{GM}{r}$$

and

$$\vec{L} = \vec{r} \times \vec{v}$$

Since $\vec{L}$ is a constant vector, both its magnitude and direction are constant. In particular, its direction is perpendicular to both $\vec{r}$ and $\vec{v}$, so that the motion of the object lies in a plane containing these vectors (the plane perpendicular to $\vec{L}$). We can therefore focus our attention to this plane only. We set up the following coordinate system



There is no natural length or time scale in the problem, so we have to rely on the initial conditions to generate these. A natural length scale is the distance r_0 of the object from the centre at $t = 0$. A time scale can be generated as follows: Imagine that the initial velocity $\vec{v}_0$ is such that the object moves along a circle with uniform speed. This will happen when

$$\begin{aligned} \frac{v_0^2}{r_0} &= \frac{GM}{r_0^2} \\ \Longrightarrow v_0 &= \sqrt{\frac{GM}{r_0}} \end{aligned}$$

Then a natural time scale is the time it would take the object to move once around the orbit. We take a time scale proportional to this as

$$T_0 = \frac{r_0}{v_0}$$

Expressing position in units of r_0, time in units of T_0 and velocity in units of v_0, we get (for convenience we use the same symbols for the dimensionaless quantities as the original ones)

$$
\begin{aligned}
\frac{d\vec{r}}{dt} &= \vec{v} \\
\frac{d\vec{v}}{dt} &= -\frac{1}{r^3}\,\vec{r}
\end{aligned}
$$

In the given coordinate system, these equations will be

$$
\begin{aligned}
\frac{dx}{dt} &= v_x \\
\frac{dv_x}{dt} &= -\frac{x}{r^3} \\
\frac{dy}{dt} &= v_y \\
\frac{dv_y}{dt} &= -\frac{y}{r^3}
\end{aligned}
$$

Classical Mechanics

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Outline

Centre of Mass

Consider a non-relativistic system of particles which interact with forces that satisfy the Third Law

$$\vec{F}_{ij} + \vec{F}_{ji} = 0$$

where $\vec{F}_{ij}$ is the force exerted by the i^{th} particle on the j^{th} particle. In absence of external forces, the total momentum of the system defined as

$$\vec{P} = \sum_i \vec{p}_i$$

is conserved

$$\begin{aligned} \frac{d\vec{P}}{dt} &= \sum_i \frac{d\vec{p}_i}{dt} \\ &= \sum_i \sum_{j \neq i} \vec{F}_{ji} \\ &= 0 \end{aligned}$$

since the forces between particles cancel in pairs.

Centre of Mass

This reminds us of the fact that the momentum of a single particle in absence of an external force is conserved. Then, the concept of momentum as an additive property allows us to 'scale' the Second Law so that its form remains unchanged for an extended system, provided we ignore the internal forces. Let $\vec{F}_i^{ext}$ be the external force acting on the i^{th} particle. The total force acting on it is then

$$\vec{F}_i^{total} = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{ji}$$

Then, is conserved

$$\begin{aligned} \frac{d\vec{P}}{dt} &= \sum_i \frac{d\vec{p}_i}{dt} \\ &= \sum_i \left(\sum_{j \neq i} \vec{F}_{ji} + \vec{F}_i^{ext} \right) \\ &= \sum_i \vec{F}_i^{ext} \end{aligned}$$

Then we get

$$\frac{d\vec{P}}{dt} = \vec{F}^{ext}$$

where $\vec{F}^{ext} = \sum_i \vec{F}_i^{ext}$ is the total external force acting on the system. This formally resembles the form of the Second Law for a sinle particle. We can extend the analogy further be *defining* a 'velocity' $\vec{V}_{cm}$ such that

$$\vec{P} = M\vec{V}_{cm}$$

such that

$$\vec{F}^{ext} = M\frac{d\vec{V}_{cm}}{dt}$$

One can further stretch the analogy by trying to imagine a 'particle' which would possess of this 'velocity'. This can be drawn from the definition of $\vec{V}_{cm}$

$$\begin{aligned}
\vec{V}_{cm} &= \frac{1}{M}\vec{P} \\
&= \frac{1}{M}\sum_i m_i \vec{v}_i \\
&= \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} \\
&= \frac{1}{M}\frac{d}{dt}\left(\sum_i m_i \vec{r}_i\right) \\
&= \frac{d\vec{R}_{cm}}{dt}
\end{aligned}$$

where

$$\vec{R}_{cm} = \frac{\sum_i m_i\, \vec{r}_i}{M}$$

This motivate us to visualise a ‘particle’ of mass M and position $\vec{R}_{cm}$. In general, none of the particles of the system is physically located at this ‘position’. However, it is useful to visualise an imaginary particle of mass M located at $\vec{R}_{cm}$. This point in space is called the *Centre of Mass*. The real advantage of this visualisation is that the generally complicated motion of the system of particles can be visualised as a combination of motion of centre of mass (under the influence of the total external force) and motion of the particles *about* the center of mass.

The Centre of Mass Frame

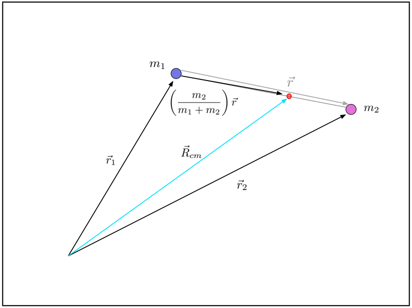
The CM frame (or zero momentum frame) is a frame of reference in which the centre of mass is at rest.

Centre of Mass of Two Particles

The position of the CM of two particles is easily seen to lie along the line joining them, between the two masses. This follows from

$$\vec{R}_{cm} = \vec{r}_1 + \frac{m_2}{m_1 + m_2}\,\vec{r}$$

Since $m_2/(m_1 + m_2) < 1$, clearly the vector $m_2/(m_1 + m_2)\vec{r}$ will be directed from m_1 to m_2 and will have length less than that of the distance between them. Then, it is easy to see from the geometry of addition of displacements that $\vec{R}_{cm}$ will lie between the positions of the two masses



Properties of CM

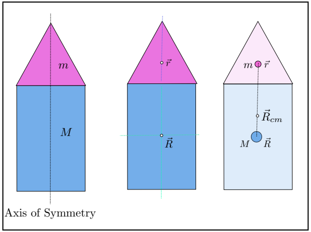
- **The position of the centre of mass satisfies the 'scaling' property.**
  Consider a system consisting of two subsystems A and B of masses M_A and M_B respectively. Let the position of their CMs be $\vec{R}_A$ and $\vec{R}_B$. Then, it is easy to see that the position of the CM of the total system satisfies

$$\vec{R}_{cm} = \frac{M_A \vec{R}_A + M_B \vec{R}_B}{M_A + M_B}$$

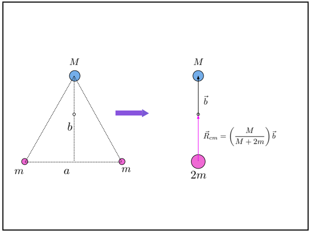
- **The CM of a system lies on every plane/axis/point of symmetry of the system**
  An plane of symmetry of a system is one about which the system 'looks the same'. That is, if one were to pretend the plane was a mirror (mirrored on both faces), the system would look the same.

Example: CM of a uniform rod of length l. Since the centre of the rod is a point of symmetry, the CM is right there!

Example: A more complicated system



Example: CM of three masses arranged at the vertices of an isoceles triangle



Note: The position of the CM relative to the particles of the system is fixed. To compute this position, any suitable origin ocan be chosen to measure position vectors. For more complicated distributions, it is useful to explicitly set up a coordinate system such that

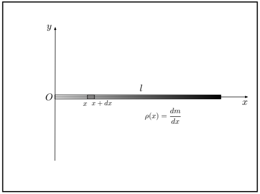
$$\begin{aligned}\vec{r}_i &= x_i\,\hat{i} + y_i\,\hat{j} + z_i\,\hat{k} \\ \vec{R}_{cm} &= X_{cm}\,\hat{i} + Y_{cm}\,\hat{j} + Z_{cm}\,\hat{k}\end{aligned}$$

Then, we get

$$\begin{aligned}X_{cm} &= \frac{\sum_i m_i\, x_i}{M} \\ Y_{cm} &= \frac{\sum_i m_i\, y_i}{M} \\ Z_{cm} &= \frac{\sum_i m_i\, z_i}{M}\end{aligned}$$

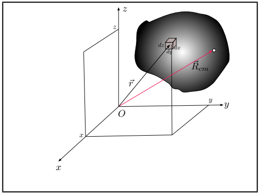
CM of Continuous Distributions

Example: CM of a rod with linear mass density $\rho(x)$.
Let us choose a convenient coordinate system as follows



$$
\begin{aligned}
X_{cm} &= \sum_x dm\ x \\
&= \sum_x \frac{dm}{dx} dx\ x \\
&= \sum_x \rho(x)\ x\ dx \\
&\rightarrow \int_0^l \rho(x)\ x\ dx
\end{aligned}
$$

General Continuous Mass Distributions



We again set up a coordinate system. Visualising the mass distribution to be divided into infinitesimal cubes (with edges along the axes) of sides dx, dy, dz, the mass contained in such a cube located at point (x, y, z) is

$$dm = \rho(x, y, z)dxdydz$$

Then

$$\begin{aligned} X_{cm} &= \sum_{x,y,z} dm(x, y, z)\ x \\ &= \sum_{x,y,z} \rho(x, y, z)\ x\ dxdydz \\ &\to \int\int\int dxdydz\ \rho(x, y, z)\ x \end{aligned}$$

Similarly,

$$Y_{cm} = \int\int\int dxdydz\ \rho(x,y,z)\ y$$

$$Z_{cm} = \int\int\int dxdydz\ \rho(x,y,z)\ z$$

Illustration: Two Particles

Consider a system of two particles (of masses m_1 and m_2) interacting with a force that is *central*, that is, depends only on their separation and their relative displacement

$$\vec{F}_{12} = f(r)\vec{r}$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$ is the relative displacement of the two particles (taken for convention from particle 1 to particle 2). Assuming there are no external forces acting on the system, the total momentum of the system will be conserved and the centre of mass will move with a uniform velocity (equal to the total momentum divided by the total mass of the system). The equations of motion for the two particles will be

$$\begin{aligned} m_1 \frac{d^2\vec{r}_1}{dt^2} &= \vec{F}_{21} \\ &= -f(r)\,\vec{r} \\ m_2 \frac{d^2\vec{r}_2}{dt^2} &= \vec{F}_{12} \\ &= f(r)\,\vec{r} \end{aligned}$$

We can eliminate the position vectors $\vec{r}_1$ and $\vec{r}_2$ in favor of the position vector of the CM $\vec{R}_{cm}$ and the relative displacement $\vec{r}$, since

$$\begin{aligned}\vec{R}_{cm} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\ \vec{r} &= \vec{r}_2 - \vec{r}_1\end{aligned}$$

This gives

$$\begin{aligned}\vec{r}_1 &= \vec{R}_{cm} - \frac{m_2}{m_1 + m_2}\vec{r} \\ \vec{r}_2 &= \vec{R}_{cm} + \frac{m_1}{m_1 + m_2}\vec{r}\end{aligned}$$

Substituting these in (either) equations for motion (and observing that $d^2\vec{R}_{cm}/dt^2 = 0$, we get

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right)\frac{d^2\vec{r}}{dt^2} = f(r)\,\vec{r}$$

We write this as

$$\mu \frac{d^2\vec{r}}{dt^2} = \vec{F}$$

where

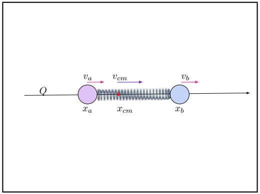
$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is called the *reduced mass* of the system and

$$\vec{F} = f(r)\,\vec{r}$$

is the internal force of interaction. This resembles the equation of motion of a *single* particle of mass μ under the influence of force $\vec{F}$.

Example: Two masses m_1 and m_2 connected by a spring, constrained to move along a straight line. In a given coordinate system, let the position of the masses at $t = 0$ be x_a and x_b and their velocity v_a and v_b



Given this, we need to determine the motion of the two masses as a function of time. The velocity of the CM will be

$$v_{cm} = \frac{m_1 v_a + m_2 v_b}{m_1 + m_2}$$

and this will be constant. At $t = 0$, the position coordinate of the CM will be

$$x_{cm}(0) = \frac{m_1 x_a + m_2 x_b}{m_1 + m_2}$$

Then, the position of the CM will change with time as

$$x_{cm}(t) = x_{cm}(0) + v_{cm}\, t$$

The position coordinates of the two masses at time t will be given by

$$\begin{aligned} x_1(t) &= x_{cm}(t) - \frac{m_2}{m_1 + m_2} x(t) \\ x_2(t) &= x_{cm}(t) + \frac{m_1}{m_1 + m_2} x(t) \end{aligned}$$

where $x(t) = x_2(t) - x_1(t)$ and satisfies the equation

$$\mu \frac{d^2 x}{dt^2} = -kx$$

where k is the spring constant and μ is the reduced mass. The general solution to this equation is

$$x(t) = A \cos \omega t + B \sin \omega t$$

where $\omega = \sqrt{k/\mu}$. The constants A and B are easilyn determined using the initial conditions $x(0) = x_b - x_a$ and $v(0) = dx/dt|_{t=0} = v_b - v_a$. Thus the general motion of the system is easily determined.

Two Body Gravitational Attraction

We analyse the motion (under gravitation) of two masses in the CM frame, so that the CM is at rest, and we choose it as origin of coordinates. At a given instant of time, the positions and velocities of the two masses are given, and the problem is to determine the trjectory of the particles. The equations of motion of the two masses are

$$\begin{aligned} m_1 \frac{d^2\vec{r}_1}{dt^2} &= \frac{Gm_1 m_2}{r^3}\,\vec{r} \\ m_2 \frac{d^2\vec{r}_2}{dt^2} &= -\frac{Gm_1 m_2}{r^3}\,\vec{r} \end{aligned}$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$. Since we have chosen the origin at the location of the CM, we get

$$\begin{aligned} \vec{r}_1 &= -\frac{m_2}{m_1 + m_2}\vec{r} \\ \vec{r}_2 &= \frac{m_1}{m_1 + m_2}\vec{r} \end{aligned}$$

such that

$$\mu \frac{d^2\vec{r}}{dt^2} = -\frac{Gm_1 m_2}{r^3}\,\vec{r}$$

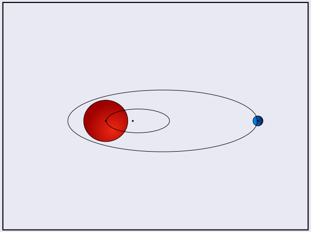
This reduces to

$$\frac{d^2\vec{r}}{dt^2} = -\frac{G\,(m_1 + m_2)}{r^3}\,\vec{r}$$

which is the equation of a particle in the presence of a gravitating object of mass $M = m_1 + m_2$.

Detection of Exoplanets

A planet of mass m orbiting a distant star of mass $M >> m$ is difficult to detect because the star's brighness will eclipse the planet. However, the fact that it is a two-body system implies that the centre of the star will move in an elliptical orbit (of dimensions much smaller than that of the planet's). If we can detect this motion of the star, not only can we predict that there is a planet, but we can also estimate its mass and orbit (provided we know the mass of the star).



Let us choose to represent the dynamics of the vector $\vec{r}$ in plane polar coordinates. Then,

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r} + \left(r\,\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\,\frac{d\theta}{dt}\right)\hat{\theta}$$

Equating this with the force, we get

$$\begin{aligned} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 &= -\frac{G\left(m_1 + m_2\right)}{r^2} \\ r\,\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\,\frac{d\theta}{dt} &= 0 \end{aligned}$$

Since the equation for $\vec{r}$ is the same as that of a particle under the influence of the gravitational influence of mass $M = m_1 + m_2$, therefore we know that the 'angular momentum' vector $\vec{L} = \vec{r} \times \vec{v}$ is conserved, where $\vec{v} = d\vec{r}/dt$

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \frac{d}{dt}\left(\vec{r} \times \vec{v}\right) \\ &= \left(\frac{d\vec{r}}{dt} \times \vec{v}\right) + \left(\vec{r} \times \frac{d\vec{v}}{dt}\right) \\ &= \left(\vec{v} \times \vec{v}\right) - \frac{G\left(m_1 + m_2\right)}{r^3}\left(\vec{r} \times \vec{r}\right) \\ &= 0 \end{aligned}$$

The direction of this vector is perpendicular to the plane of the orbit. To determine its (conserved) magnitude, we recall that $\vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$. Then,

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{v} \\ &= r\,\hat{r} \times \left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right) \\ &= (r^2\,\dot{\theta})\,\hat{r} \times \hat{\theta} \end{aligned}$$

Since $\hat{r}$ and $\hat{\theta}$ are perpendicular to each other (and each has unit magnitude), it follows that the magnitude of $\vec{L}$ is

$$L = r^2\,\dot{\theta}$$

L will be conserved and is fixed by the initial conditions. Then, we can express $\dot{\theta} = L/r^2$, so that the differential equation for r becomes

$$\ddot{r} = \frac{L^2}{r^3} - \frac{G\,(m_1 + m_2)}{r^2}$$

We can right the right hand side as derivative of a function with respect to r

$$\ddot{r} = \frac{dg}{dr}$$

where

$$g(r) = \frac{-L^2}{2r^2} + \frac{GM}{r}$$

Multiplying both sides with $\dot{r}$, we get

$$\dot{r}\,\ddot{r} = \frac{dg}{dr}\,\dot{r}$$

It is easy to check that the left hand side is the derivative of $\dot{r}^2/2$ and the right hand side is dg/dt. Then, we get

$$\frac{d}{dt}\frac{\dot{r}^2}{2} = \frac{dg}{dt}$$

which tells us that

$$\frac{d}{dt}\left(\frac{\dot{r}^2}{2} - g(r)\right) = 0$$

Therefore, we see that in addition to the angular momentum, we have another conserved quantity (which we will see is the mechanical energy of the two particle system)

$$E = \frac{1}{2}\dot{r}^2 + \frac{L^2}{2r^2} - \frac{GM}{r}$$

It is easily checked that the quantity $\dot{r}^2/2 + L^2/2r^2$ is just $\vec{v}^2/2$, which is just the 'kinetic energy'.

We will see that the solution to the radial equation yields different kinds of orbits for the two masses, depending on the values of E and L. In particular, for special initial condition, each mass will execute a circular orbit about the centre of mass. this will clearly happen when r does not change. Setting $\ddot{r} = 0$ in the radial equation of motion, we get

$$\begin{aligned} \frac{L^2}{r_0^3} &= \frac{GM}{r_0^2} \\ \Longrightarrow L &= \sqrt{G\,M\,r_0} \end{aligned}$$

For a circular orbit, the energy of the system is given by

$$\begin{aligned} E &= \frac{L^2}{2r_0^2} - \frac{GM}{r_0} \\ &= -\frac{GM}{2r_0} \end{aligned}$$

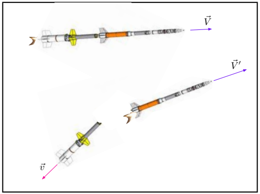
With the origin at the CM, the position vectors of the two masses are

$$\vec{r}_1 = -\frac{m_2}{m_1 + m_2}\vec{r}$$
$$\vec{r}_2 = \frac{m_1}{m_1 + m_2}\vec{r}$$

For initial condition such that $r = r_0$ is constant, clearly r_1 and r_2 will be constant, so that the two masses will move in circular orbits about the CM. Further, since the CM is along the line joining them, they will move in their orbits with the same angular velocity $\dot{\theta} = L/r_0^2$ and therefore they will revolve about the common centre of mass with the same time period.

Variable Mass Systems

A rocket of mass M is moving with velocity $\vec{V}$. At a certain instant of time it ejects a payload of mass m with velocity $\vec{u}_{rel}$ relative to itself. What is the velocity of the rocket after this?



Let the velocity of the rocket after ejection be $\vec{V}'$. Then, in the frame in which the rocket had initial velocity $\vec{V}$, the velocity of the payload will be

$$\vec{v} = \vec{V}' + \vec{u}_{rel}$$

The initial momentum of the system was $\vec{P} = M\vec{V}$. The final momentum of the system will be

$$\vec{P}' = m\vec{v} + (M - m)\vec{V}'$$

Assuming the ejection takes very little time so that any external force (such as gravity) has very little time to change the total momentum of the system (in comparison to the momentum of either object), the momentum of the system will be (approximately) conserved. Then, $\vec{P'} = \vec{P}$ (in this approximation), so that

$$M\vec{V} = m(\vec{V}' + \vec{u}_{rel}) + (M - m)\vec{V}'$$

This can be solved for $\vec{V}'$ to give

$$\vec{V}' = \vec{V} - \frac{m}{M}\,\vec{u}_{rel}$$

Rocket Motion

A rocket flies by ejecting 'hot', energetic matter (gases, typically). Since these are ejected out with some relative velocity, they carry away momentum, changing the momentum of the rocket.

Let the mass of the rocket at instant t be $M(t)$ and its velocity $\vec{V}(t)$. In time Δt, it ejects out mass Δm of hot gases with relative velocity $\vec{u}_r el$. In addition, let some external force (typically the force due to gravity) $\vec{F}$ act on the rocket, and at time $t + \Delta t$, let its velocity be $\vec{V}(t) + \Delta\vec{V}$. Then, the change in momentum of the rocket in time Δt will be

$$\begin{aligned}
\Delta\vec{P} &= \vec{P}(t+\Delta t) - \vec{P}(t) \\
&= \left\{\Delta m\left(\vec{V}(t) + \Delta\vec{V} + \vec{u}_{rel}\right) + (M(t) - \Delta m)\left(\vec{V}(t) + \Delta\vec{V}\right)\right\} - M(t)\vec{V}(t) \\
&= \Delta m\,\vec{u}_{rel} + M(t)\Delta\vec{V}
\end{aligned}$$

where terms of the type $\Delta m\Delta\vec{V}$ have been dropped since they are second order in infinitesimal quantities.

Then the rate of change of momentum of the system is

$$\begin{aligned}\frac{d\vec{P}}{dt} &= \lim_{\Delta t \to 0} \frac{\Delta \vec{P}}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t}\, \vec{u}_{rel} + M(t)\, \frac{\Delta \vec{V}}{\Delta t} \\ &= \frac{dm}{dt}\, \vec{u}_{rel} + M(t)\, \frac{d\vec{V}}{dt}\end{aligned}$$

Since the total mass is conserved, the mass of the fuel ejected in time Δt is equal to the decrease in mass of the rocket in the same interval. That is, $\Delta m = -\Delta M$ where ΔM is the change in mass of the rocket in time Δt. Then,

$$\frac{d\vec{P}}{dt} = -\frac{dM}{dt}\, \vec{u}_{rel} + M(t)\, \frac{d\vec{V}}{dt}$$

Since this is equal to the external force acting on the system, we get the 'Rocket Equation'

Rocket Equation

$$M(t)\, \frac{d\vec{V}}{dt} - \frac{dM}{dt}\, \vec{u}_{rel} = \vec{F}$$

Rocket in Free Space

Consider a rocket far away from gravitating objects. Then, $\vec{F} = 0$ and the rocket equation becomes

$$M(t)\,\frac{d\vec{V}}{dt} = \frac{dM}{dt}\,\vec{u}_{rel}$$

Assuming that the gases are ejected out in a direction along the length of the rocket, the motion of the rocket will be rectilinear. Choosing coordinates such that the motion of the rocket is along the x-direction, we have $\vec{V} = V\hat{i}$ and $\vec{u}_{rel} = -u$ where we assume that the gases are ejcted with the same relative speed u. Then, we have

$$M(t)\,\frac{dV}{dt} = \frac{dM}{dt}u$$

$$\implies dV = \frac{dM}{M}u$$

where dV is the change in speed of the rocket in time dt and dM is the change in its mass.

Integrating from an instant when its speed was V_0 and its mass was M_0 to an instant t when its speed is $V(t)$ and mass is $M(t)$, we get

$$\int_{V_0}^{V(t)} dV = u \int_{M_0}^{M(t)} \frac{dM}{M}$$

which gives

Rocket in Free Space

$$V(t) - V_0 = u \log\left(\frac{M(t)}{M_0}\right)$$

This equation has an interesting implication: so long as the fuel is burnt up and ejected with the same relative velocity, the maximum increase in speed of a rocket depends only on the *total* mass of the fuel ejected, and not on the rate at which it is ejected. Of course, the total time to achieve this speed will depend on this rate.

Rocket at Time of Launch

At the moment a rocket is launched, it is under the influence of the Earth's gravitational field (assumed uniform here). Setting up a coordinate system with the vertical direction (along which we assumed the rocket is launched) as the x-direction, the rocket equation reduces to

$$M(t)\,\frac{dV}{dt} - \frac{dM}{dt}\,u = -M(t)\,g$$

This reduces to

$$dV - u\,\frac{dM}{M} = -g\,dt$$

for the infinitesimal duration of time dt. Integrating once again from $t = 0$ (at which the speed of the rocket is zero) to t, we get

$$\int_0^{V(t)} dV \;-\; u \int_{M_0}^{M(t)} \frac{dM}{M} = -g \int_0^t dt$$

which gives

Rocket at Launch

$$V(t) = u\,\log\left(\frac{M(t)}{M_0}\right) - g\,t$$

It is clear that the maximum possible speed that the rocket can acquire is limited now by the rate at which the fuel is ejected. To minimise the effect of gravity, the duration t needs to be minimised. That is, the fuel needs to be expended as fast as possible for an effective launch.

Classical Mechanics

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Outline

Consider a system of two particles constrained to move along a straight line interacting with a force that depends only on their relative separation. The equations of motion for the two particles will be then

$$m_1 \frac{dv_1}{dt} = f(x)$$
$$m_2 \frac{dv_2}{dt} = -f(x)$$

where $x = x_2 - x_1$, $F_{12} = -F_{21} = f(x)$. Multiplying the first equation with v_1 and the second with v_2 and adding, we get

$$m_1 v_1 \frac{dv_1}{dt} + m_2 v_2 \frac{dv_2}{dt} = f(x)\,(v_1 - v_2)$$
$$= -f(x)\,\frac{dx}{dt}$$

where $v_2 - v_1 = d(x_2 - x_1)/dt = dx/dt$.

We can always find a function $U(x)$ such that

$$f(x) = \frac{dU(x)}{dx}$$

Observing that

$$\begin{aligned} v_1 \frac{dv_1}{dt} &= \frac{1}{2}\frac{dv_1^2}{dt} \\ v_2 \frac{dv_2}{dt} &= \frac{1}{2}\frac{dv_2^2}{dt} \end{aligned}$$

we get

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right) &= -\frac{dU}{dx}\frac{dx}{dt} \\ &= -\frac{dU}{dt} \end{aligned}$$

This allows us to define a quantity E (which we call *energy*)

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + U(x_2 - x_1)$$

which is *conserved*

$$\frac{dE}{dt} = 0$$

The quantity

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

is called the *Kinetic energy* of the system and the quantity $U(x_2 - x_1)$ is called the *Potential Energy* of the system. The Kinetic Energy of the system is by virtue of its motion and the potential energy by virtue of the relative positions of the particles. We have already observed that the total momentum of this system is conserved, where

$$P = m_1 v_1 + m_2 v_2$$

Then, we have two conserved quantities.

Now, assume that one of the particles is much more massive compared with the other, say, $m_1 >> m_2$. Assume that the CM is at rest (we are in such an inertial frame). Then $P = 0$ and

$$\begin{aligned} v_1 &= -\frac{m_2}{m_1} v_2 \\ &\approx 0 \end{aligned}$$

Since the CM will lie approximately at the location of the massive particle, let us choose the origin at the CM. Then, $x_2 = x$ and $v_2 = dx/dt$. The total energy of the system now becomes

$$\begin{aligned} E &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + U(x_2 - x_1) \\ &\approx \frac{1}{2} m_2 v_2^2 + U(x_2) \end{aligned}$$

where the force experienced by m_2 is

$$\begin{aligned} F_2 &= -f(x) \\ &= -\frac{dU}{dx} \\ &= -\frac{dU}{dx_2} \end{aligned}$$

Since the dynamics of m_1 is irrelevant (its position approximately does not change with time), *effectively*, we can attribute the energy E to m_2 only, so that we say that the energy of m_2 is conserved. In such a situation, we can take an equivalent point of view. We can say that m_1 exerts a *position dependent* force on m_2, that is, one that depends only on the position of m_2, which can therefore be written as the derivative of a *Potential Energy function*

$$F(x) = -\frac{dU(x)}{dx}$$

where x is the position of m_2 (relative to the unchanging position of m_1). Then, the Second Law of motion applied to m_2 is

$$\begin{aligned} m\frac{dv}{dt} &= F(x) \\ &= -\frac{dU}{dx} \end{aligned}$$

Multiplying both sides with v

$$F(x)\frac{dx}{dt} = mv\frac{dv}{dt}$$

In an interval dt, we then have

$$F(x)\ dx = m\ v\ dv$$

Integrating from time t_1 (when position is x_1 and velocity v_1) to time t_2 (when position is x_2 and velocity v_2)

$$\begin{aligned}\int_{x_1}^{x_2} dx\ F(x) &= \int_{v_1}^{v_2} m\ v\ dv \\ &= \frac{m}{2}\int_{v_1}^{v_2} d\left(v^2\right) \\ &= \frac{1}{2}m\ v_2^2 - \frac{1}{2}m\ v_1^2\end{aligned}$$

The integral

$$W = \int_{x_1}^{x_2} dx\ F$$

is called the *Work done by force F* on the prticle as it moves from x_1 to x_1. Note that even if the force on the particle in not just a function of position (and depends on its velocity, time, etc.), we still define this integrtal as *Work*. Then, the above result is called *The Work-energy Theorem*, which says that the work done by a force on a particle equals the change in its kinetic energy. Note that this result is valid even if the force acting on the particle is not a function of position alone, and is simply a consequence of the Second Law of Motion. However, if the force on the particle is a function of its position, the Work done by the force is seen to be equal to (minus) the change in the potential energy function

$$\begin{aligned} \int_{x_1}^{x_2} dx\ F(x) &= -\int_{x_1}^{x_2} dx\ \frac{dU}{dx} \\ &= --\int_{x_1}^{x_2} dU \\ &= -(U(x_2) - U(x_1)) \end{aligned}$$

Combining this result with th Work-Energy theorem, we recover the result for conservation of energy

$$\begin{aligned} \frac{1}{2}m\,v_2^2 - \frac{1}{2}m\,v_1^2 &= -\left(U(x_2) - U(x_1)\right) \\ \implies \frac{1}{2}m\,v_2^2 + U(x_2) &= \frac{1}{2}m\,v_1^2 + U(x_1) \end{aligned}$$

Let us now go to the three-dimensional situation where the force between the two particles is of the form

$$\vec{F}_{12} = -\vec{F}_{21} = f(r)\,\hat{r}$$

where r is the (instantaneous) distance between the two particles and $\vec{r} = \vec{r}_2 - \vec{r}_1$ is the relative displacement vector. The unit vector $\hat{r}$ is

$$\hat{r} = \left(\frac{1}{r}\right)\vec{r}$$

The equations of motion for the two particles are

$$\begin{aligned} m_1 \frac{d\vec{v}_1}{dt} &= f(r)\,\hat{r} \\ m_2 \frac{d\vec{v}_2}{dt} &= -f(r)\,\hat{r} \end{aligned}$$

Taking the dot product of the first equation with $\vec{v}_1$, the second equation with $\vec{v}_2$ and adding, we get

$$\begin{aligned} m_1 \vec{v}_1 \cdot \frac{d\vec{v}_1}{dt} + m_2 \vec{v}_2 \frac{d\vec{v}_2}{dt} &= f(r)\left(\vec{v}_1 - \vec{v}_2\right) \cdot \hat{r} \\ &= -f(r)\,\hat{r} \cdot \frac{d\vec{r}}{dt} \end{aligned}$$

where $\vec{v}_2 - \vec{v}_1 = d(\vec{r}_2 - \vec{r}_1)/dt = d\vec{r}/dt$. We use the result

$$\frac{d}{dt}\left(\vec{A} \cdot \vec{A}\right) = 2\vec{A} \cdot \frac{d\vec{A}}{dt}$$

Then the left hand side of the equation is just dK/dt where we have defined

$$\begin{aligned} K &= \frac{1}{2} m_1 \vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2} m_2 \vec{v}_2 \cdot \vec{v}_2 \\ &= \frac{1}{2} m_1 \vec{v}_1^2 + \frac{1}{2} m_2 \vec{v}_2^2 \end{aligned}$$

as the total *Kinetic Energy* of the system. We can always find a function $U(r)$ such that

$$f(r) = \frac{dU}{dr}$$

Then the right hand side of the equation becomes

$$\begin{aligned} -f(r)\,\hat{r} \cdot \frac{d\vec{r}}{dt} &= -\frac{dU}{dr}\,\hat{r} \cdot \frac{d\vec{r}}{dt} \\ &= -\vec{\nabla} U \cdot \frac{d\vec{r}}{dt} \\ &= -\frac{dU}{dt} \end{aligned}$$

Then, we get

$$\begin{aligned} \frac{dK}{dt} &= -\frac{dU}{dt} \\ \Longrightarrow \frac{dE}{dt} &= 0 \end{aligned}$$

where

$$E = \frac{1}{2} m_1 \vec{v}_1^2 + \frac{1}{2} m_2 \vec{v}_2^2 + U(r)$$

Then E, a conserved quantity, is identified as the total Energy of the system. As before, the total momentum of the system

$$\vec{P} = m_1 \vec{v}_1 + m_2 \vec{v}_2$$

is conserved in addition to energy.

As before, if one of the particles is very massive compared with the other ($m_1 >> m_2$) then in the CM frame, $\vec{v}_1 \approx 0$ so that the dynamics of the massive particle are irrelevant. In this approximation, choosing the origin at the CM (the location of the massive particle), the total energy of the system is

$$E = \frac{1}{2} m_2 \vec{v}_2^2 + U(r_2)$$

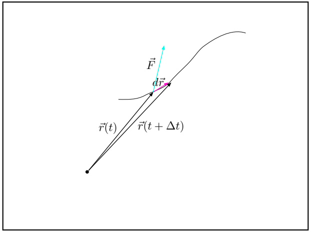
where $\vec{r}_2$ is the position vector of m_2. The force experienced by m_2 is

$$\begin{aligned} \vec{F}_2 &= -f(r_2)\hat{r}_2 \\ &= -\frac{dU}{dr_2}\,\hat{r}_2 \\ &= -\vec{\nabla}_2 U(r_2) \end{aligned}$$

As before, since the dynamics of m_1 are irrelevant, we can take the following point of view: mass m_2 moves under the influence of m_1 (whose position is stationary) such that the equation of motion is

$$\begin{aligned} m\frac{d\vec{v}}{dt} &= \vec{F} \\ &= -\vec{\nabla}U(r) \end{aligned}$$

where we have dropped the subscript 2. Clearly $\vec{F}$ is only a function of the position $\vec{r}$ of the particle. We now define the *Work done by* $\vec{F}$ on the particle during its motion from time t_1 to t_2 as follows: during this interval, the particle follows a trajectory. We divide the duration $t_2 - t_1$ into N intervals of size δt. Let $d\vec{r}$ be the infinitesimal displacement of the particle from instant t to $t + \Delta t$



Let the force acting on the particle when it is located at $\vec{r}$ be $\vec{F}$. We take the dot product of this force with the infinitesimal displacemnt at every instant $t_n = t_i + n\Delta t$ and add up the contributions. The Work done by the force is this sum in the limit $N \to \infty$

$$
\begin{aligned}
W &= \sum_{n=0}^{N-1} \vec{F}\left(\vec{r}_n\right) \cdot d\vec{r} \\
&= \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}\left(\vec{r}\right) \cdot d\vec{r}
\end{aligned}
$$

Using the second Law of Motion, we get

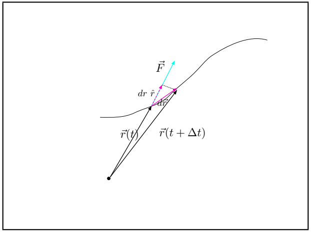
$$\begin{aligned}
W &= \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}(\vec{r}) \cdot d\vec{r} \\
&= m \int_{\vec{r}_i}^{\vec{r}_f} \frac{d\vec{v}}{dt} \cdot d\vec{r} \\
&= m \int_{\vec{r}_i}^{\vec{r}_f} \frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt}\, dt \\
&= m \int_{t_i}^{t_f} \frac{d\vec{v}}{dt} \cdot \vec{v} dt \\
&= \frac{1}{2} m \int_{t_i}^{t_f} \frac{d}{dt} (\vec{v} \cdot \vec{v})\, dt \\
&= \frac{1}{2} m \int_{t_i}^{t_f} \frac{d}{dt} \left(\vec{v}^2\right) dt \\
&= \frac{1}{2} m \left(\vec{v}_2^2 - \vec{v}_1^2\right) \\
&= \frac{1}{2} m\, \vec{v}_2^2 - \frac{1}{2} m\, \vec{v}_1^2 \\
&= K_2 - K_1
\end{aligned}$$

This is the Work-Energy Theorem, and is valid irrespective of whether the external force depends on position or not. Coming back to the example of a particle under the influence of force due to a much more massive particle (a force which is central), we know that

$$\vec{F} = -\vec{\nabla} U(r)$$

Then, the Work done by the force is also equal to

$$\begin{aligned} W &= \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}\left(\vec{r}\right) \cdot d\vec{r} \\ &= -\int_{\vec{r}_i}^{\vec{r}_f} \vec{\nabla} U(r) \cdot d\vec{r} \\ &= -\int_{\vec{r}_i}^{\vec{r}_f} \frac{dU}{dr}\, \hat{r} \cdot d\vec{r} \end{aligned}$$



The illustration shows that the dot product $\hat{r} \cdot d\vec{r}$ is just equal to dr (the change in the radial coordinate of the particle as it moves from $\vec{r}$ to $\vec{r} + d\vec{r}$. Then, the work is given by

$$\begin{aligned} W &= -\int_{r_i}^{r_f} \frac{dU}{dr}\, dr \\ &= -\int_{r_i}^{r_f} dU \\ &= -(\,U(r_2) - U(r_1)\,) \end{aligned}$$

Equating the two expressions for Work, we get

$$\begin{aligned} K_2 - K_1 &= -(\,U(r_2) - U(r_1)\,) \\ \Longrightarrow K_2 + U(r_2) &= K_1 + U(r_1) \end{aligned}$$

which is just the expression for conservation of energy.

Now, consider a system of N particles with the force exerted by the i^{th} particle on the j^{th} particle given by

$$\vec{F}_{ij} = f_{ij}(r_{ij})\,\hat{r}_{ij}$$

where r_{ij} is the distance between the particles and $\hat{r}_{ij}$ is a unit vector directed from the position of the j^{th} particle to the i^{th} particle. Then the equations of motion for these particles are

$$\begin{aligned} m_i \frac{d\vec{v}_i}{dt} &= \sum_{j \neq i} \vec{F}_{ji} \\ &= \sum_{j \neq i} f_{ji}(r_{ji})\,\hat{r}_{ji} \end{aligned}$$

Taking the dot product with $\vec{v}_i$ and summing over all i, we get

$$\sum_i m_i \vec{v}_i \cdot \frac{d\vec{v}_i}{dt} = \sum_i \sum_{j \neq i} f_{ji}(r_{ji})\,\hat{r}_{ji} \cdot \vec{v}_i$$

It is easy to convince oneself that the sum on the right will split into pairs of terms such as $f_{ji}(r_{ji})\,\hat{r}_{ji}\cdot\vec{v}_i + f_{ij}(r_{ij})\,\hat{r}_{ij}\cdot\vec{v}_j$ where $f_{ji}(r_{ji}) = f_{ij}(r_{ij})$ (This is consistent with the Third Law since $\hat{r}_{ji} = -\hat{r}_{ij}$). Further, the left hand side is equal to

$$\begin{aligned}\sum_i m_i\vec{v}_i\cdot\frac{d\vec{v}_i}{dt} &= \sum_i \frac{1}{2}m_i\frac{d\left(\vec{v}_i\cdot\vec{v}_i\right)}{dt}\\ &= \sum_i \frac{1}{2}m_i\frac{dv_i^2}{dt}\\ &= \frac{dK}{dt}\end{aligned}$$

where

$$\begin{aligned}K &= \sum_i \frac{1}{2}m_i v_i^2\\ &= \sum_i K_i\end{aligned}$$

is the total Kinetic Energy of the system.

Then, we have

$$\begin{aligned}\frac{dK}{dt} &= \sum_{pairs} \left[f_{ji}(r_{ji})\,\hat{r}_{ji}\cdot\vec{v}_i + f_{ij}(r_{ij})\,\hat{r}_{ij}\cdot\vec{v}_j\right] \\ &= \sum_{pairs} \left[f_{ij}(r_{ij})\,\hat{r}_{ij}\cdot(\vec{v}_j - \vec{v}_i)\right] \\ &= \sum_{pairs} \left[-f_{ij}(r_{ij})\,\hat{r}_{ij}\cdot\frac{d\vec{r}_{ij}}{dt}\right]\end{aligned}$$

where $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ and $\hat{r}_{ij} = \vec{r}_{ij}/r_{ij}$. We can always find functions $U_{ij}(r_{ij})$ such that

$$f_{ij}(r_{ij}) = \frac{dU_{ij}(r_{ij})}{dr_{ij}}$$

Then

$$\begin{aligned}\frac{dK}{dt} &= \sum_{pairs}\left[-\frac{dU_{ij}(r_{ij})}{dr_{ij}}\,\hat{r}_{ij}\cdot\frac{d\vec{r}_{ij}}{dt}\right] \\ &= \sum_{pairs}\left[-\frac{dU_{ij}(r_{ij})}{dr_{ij}}\,\frac{dr_{ij}}{dt}\right] \\ &= \sum_{pairs}\left[-\frac{dU_{ij}(r_{ij})}{dt}\right] \\ &= -\frac{d}{dt}\sum_{pairs}U_{ij}(r_{ij})\end{aligned}$$

where we have used the fact that $\hat{r}_{ij}\cdot d\vec{r}_{ij}/dt = dr_{ij}/dt$ since only the radial displacement is relevant to the dot product. Then, we finally get

$$\frac{d}{dt}\left[\sum_i K_i + \sum_{pairs}U_{ij}(r_{ij})\right] = 0$$

which tells us that there exists a conserved energy, given by

$$E = \sum_i K_i + \sum_{pairs}U_{ij}(r_{ij})$$

We identify the term

$$U = \sum_{pairs} U_{ij}(r_{ij})$$

as the potential energy of the system of particles. Note that potential energy cannot be written in a form that allows us to interpret it as a sum of potential energies of the individual particles. It is a *shared* property of the system of particles. However, from this single expression, the force acting on any particle can be easily extracted.

To see this, consider the force acting on the i^{th} particle. This is given by

$$
\begin{aligned}
\vec{F}_i &= \sum_{j \neq i} \vec{F}_{ji} \\
&= \sum_{j \neq i} f_{ji}(r_{ji})\, \hat{r}_{ji} \\
&= \sum_{j \neq i} \frac{dU_{ji}(r_{ji})}{dr_{ji}} \hat{r}_{ji} \\
&= -\sum_{j \neq i} \frac{dU_{ij}(r_{ij})}{dr_{ij}} \hat{r}_{ij} \\
&= -\sum_{j \neq i} \vec{\nabla}_{ij} U_{ij}(r_{ij})
\end{aligned}
$$

where the gradient $\vec{\nabla}_{ij}$ is defined as

$$
\vec{\nabla}_{ij} g = \frac{\partial g}{\partial x_{ij}}\, \hat{i} + \frac{\partial g}{\partial y_{ij}}\, \hat{j} + \frac{\partial g}{\partial z_{ij}}\, \hat{k}
$$

with $x_{ij} = x_i - x_j$ etc. and where g is any function of x_{ij}, y_{ij}, z_{ij}.

It is easy to check that

$$\frac{\partial g}{\partial x_i} = -\frac{\partial g}{\partial x_j} = \frac{\partial g}{\partial x_{ij}}$$

etc. so that

$$\begin{aligned}
\vec{\nabla}_{ij} g &= \frac{\partial g}{\partial x_{ij}}\,\hat{i} + \frac{\partial g}{\partial y_{ij}}\,\hat{j} + \frac{\partial g}{\partial z_{ij}}\,\hat{k} \\
&= \frac{\partial g}{\partial x_i}\,\hat{i} + \frac{\partial g}{\partial y_i}\,\hat{j} + \frac{\partial g}{\partial z_i}\,\hat{k} \\
&= \vec{\nabla}_i g
\end{aligned}$$

and similarly

$$\vec{\nabla}_{ij} g = -\vec{\nabla}_j g$$

Then it follows that the force acting on the i^{th} particle is

$$\begin{aligned}
\vec{F}_i &= -\sum_{j\neq i} \vec{\nabla}_{ij} U_{ij}(r_{ij}) \\
&= -\sum_{j\neq i} \vec{\nabla}_i U_{ij}(r_{ij}) \\
&= -\vec{\nabla}_i \sum_{j\neq i} U_{ij}(r_{ij})
\end{aligned}$$

Since the only terms that depend on x_i, y_i and z_i in the expression for the total potential energy of the system $U = \sum_{pairs} U_{ij}(r_{ij})$ are the those that come in the combination $\sum_{j \neq i} U_{ij}(r_{ij})$ therefore we can finally write

$$\vec{F}_i = -\vec{\nabla}_i U$$

which tells us that we can extract the force acting on any particle by calculating the gradient of the total potential energy with respect to coordinates of that particle. Now let us imagine that one of the particles, say particle with mass m_1 is much more massive than the others. First, we choose an inertial frame in which the CM of the system is at rest. The total momentum of the system in this frame is zero. Then it follows that the velocity of the massive particle will be

$$\begin{aligned} \vec{v}_1 &= -\frac{m_2}{m_1}\vec{v}_2 - \frac{m_3}{m_1}\vec{v}_3 - .... - -\frac{m_N}{m_1}\vec{v}_N \\ &= \approx 0 \end{aligned}$$

Then the dynamics of this particle are irrelevant. In particular, its contribution to the total kinetic energy is vanishingly small, since it is

$$\frac{\vec{p}_1^2}{2m_1} \approx 0$$

Then the total energy of the system will be approximately

$$E \approx \frac{\vec{p}_2^2}{2m_2} + .. + \frac{\vec{p}_N^2}{2m_N} + U$$

Further, since the coordinates of this particle do not change with time, the potential energy of the system can be thought of as a function of position coordinates of the rest of the particles only. Then, we may safely attribute this entire energy to the rest of the $N-1$ particles. Note however that the momentum of these particles will not be conserved, since the massive particle, even though its position does not change, does exert forces on the rest of the particles. Note that this argument in general *cannot* be extended to a situation in which more than one particle is much more massive than the rest. For instance, say there are three particles, with two having mass M and the third mass m with $M >> m$. Then in the CM frame,

$$M\vec{v}_1 + M\vec{v}_2 + m\vec{v}_3 = 0$$

Then we have

$$\vec{v}_1 = -\vec{v}_2 - \frac{m}{M}\vec{v}_3$$

which clearly does not imply that either $\vec{v}_1$ or $\vec{v}_2$ should be vanishingly small. Then, in general, if we have a system of more than two particles, there is no, in general, even approximately, a concept of conservation of energy o a sub-system, even if the rest of the system is much more massive than the sub-system.

Conservative and Non-Conservative Forces

Force acting on a particle is said to be conservative if it allows for the existence of a conserved energy for the particle. That is, one can associate a Potential energy function $U(\vec{r})$ with the interaction the force represents, such that

$$E = \frac{1}{2} m\, \vec{v}^2 + U(\vec{r})$$

is conserved. Let us determine the necessary and sufficient condition under which a force is conservative. We start with the Work-Energy theorem. Assume that the particle moves from point $\vec{r}_1$ to point $\vec{r}_2$ under the influence of the force. Then the work done by the force as t moves from $\vec{r}_1$ to $\vec{r}_2$ is given by

$$W = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}$$

In general, this work depends on the details of the trajectory taken by the particle from $\vec{r}_1$ to $\vec{r}_2$. For instance, the force due to friction (assuming it is approximately given by $F = \mu N$ where μ is the coefficient of friction and N is the normal force) opposes motion. Therefore, if the particle moves through a small displacement $d\vec{r}$, the work done by friction will be

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{r} \\ &= -\left|\vec{F}\right| |d\vec{r}| \\ &= -\mu N \, dl \end{aligned}$$

where dl is the length of the small path segment. Clearly, dW will always be a negative number. This implies that the more the distance covered by the particle in going from $\vec{r}_1$ to $\vec{r}_2$, the more (negative) will be the work done. Assuming for a moment that the normal force remains constant along the motion, the total work done will be

$$W = -\mu N \, l$$

where l is the total length of the path in going from $\vec{r}_1$ to $\vec{r}_2$. Clearly, the work done by the force of friction depends on the details of the trajectory taking the particle from one point to another. Similarly, work done by a viscous force (proportional to velocity and directed opposite to it) will always be negative and will depend on the details of the trajectory. As another example, consider a particle moving along a straight line (along the x-axis). If the force acting on the particle depends only on the position of the particle, then it can alwas be expressed as a derivative of another function of position

$$F(x) = -\frac{dU(x)}{dx}$$

Then the work done by the force as the particle goes from x_1 to x_2 will be

$$\begin{aligned} W &= \int_{x_1}^{x_2} dx\, F(x) \\ &= -\int_{x_1}^{x_2} dx \frac{dU(x)}{dx} \\ &= -\int_{x_1}^{x_2} dx dU \\ &= U(x_1) - U(x_2) \end{aligned}$$

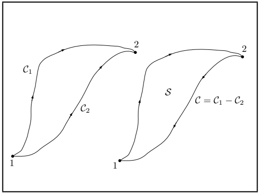
which depends only on the end-points and not on the details of the trajectory connecting them. However, if the force depnds in addition on time or velocity of the particle or both, the work done *will* depend on the details of the trajectory.

In case of a particle moving in three dimensions, it is not sufficient for the force to be only a function of position for the work to be independent of the path.

Condition for Path Independence of Work

The necessary and sufficient condition for the work done by a force to be independent of the path is that its curl is zero.

Proof:



Consider two paths connecting points 1 and 2 (paths C_1 and C_2). If the work done by the force is independent of the path, then

$$\int_{\mathcal{C}_1} d\vec{r} \cdot \vec{F} = \int_{\mathcal{C}_2} d\vec{r} \cdot \vec{F}$$

This implies that $W_1 - W_2 = 0$ or equivalently

$$\oint_{\mathcal{C}} d\vec{r} \cdot \vec{F} = 0$$

where $\mathcal{C}$ is the closed loop path defined by $\mathcal{C} = \mathcal{C}_1 - \mathcal{C}_2$.

It follows from Stokes Theorem that

$$\oint_{\mathcal{C}} d\vec{r} \cdot \vec{F} = \iint_{\mathcal{S}} d\vec{A} \cdot (\vec{\nabla} \times \vec{F})$$

where $\mathcal{S}$ is the surface bound by the loop. Since the work is independent of the path, we can tke points 1 and 2 arbitrarily close and the loop C arbitrarily small. for a very tiny loop, the surface integral will be

$$\iint_{\mathcal{S}} d\vec{A} \cdot (\vec{\nabla} \times \vec{F}) = \Delta A \, \hat{n} \cdot (\vec{\nabla} \times \vec{F})$$

where ΔA is the area of the loop and $\hat{n}$ is a unit vector normal to the area. Since This is true for any loop, the unit vector $\hat{n}$ can have any direction. Then it follows that

$$\hat{n} \cdot (\vec{\nabla} \times \vec{F}) = 0$$

for arbitrary $\hat{n}$. Then

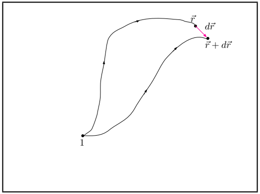
$$\vec{\nabla} \times \vec{F} = 0$$

Clearly, the converse is true. That is, if $\vec{\nabla} \times \vec{F} = 0$, the work done is independent of the path.

An equivalent statement is that the force can be expressed as the gradient of a function (which is called the Potential Energy Function). It is clear that if this is true then the curl of the force is zero. To see the converse, let us take a fixed reference point $\vec{r}_0$ and compute the work done by $\vec{F}$ along an arbitrary path from $\vec{r}_0$ to $\vec{r}$. Since this is independent of the path, for a given $\vec{r}_0$, it can only depend on the final point $\vec{r}$. Then, we can define a function $U(\vec{r})$ such that

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} d\vec{r} \cdot \vec{F}$$

where the negative sign is conventional. This clearly tells us that the function so defined is such that $U(\vec{r}_0) = 0$. Now, consider the change in U as the final point is changed from $\vec{r}$ to $\vec{r} + d\vec{r}$.



Then

$$dU = -\left(\int_{\vec{r}_0}^{\vec{r}+d\vec{r}} d\vec{r}\cdot\vec{F} - \int_{\vec{r}_0}^{\vec{r}} d\vec{r}\cdot\vec{F}\right)$$

Since the work done is independent of the path, we can choose the path connecting $\vec{r}_0$ to $\vec{r}+d\vec{r}$ to be such that it coincides with the first path till the point $\vec{r}$ and then goes to the point $\vec{r}+d\vec{r}$ along an infinitesimal straight segment $d\vec{r}$. Then,

$$\begin{aligned}\int_{\vec{r}_0}^{\vec{r}+d\vec{r}} d\vec{r}\cdot\vec{F} &= \int_{\vec{r}_0}^{\vec{r}} d\vec{r}\cdot\vec{F} + \int_{\vec{r}}^{\vec{r}+d\vec{r}} d\vec{r}\cdot\vec{F}\\ &\approx \int_{\vec{r}_0}^{\vec{r}} d\vec{r}\cdot\vec{F} + d\vec{r}\cdot\vec{F}\end{aligned}$$

Using the result

$$dU = \vec{\nabla}U\cdot d\vec{r}$$

we finally get

$$\vec{\nabla}U\cdot d\vec{r} = -\vec{F}\cdot d\vec{r}$$

Since this is true for arbitrary $d\vec{r}$, it follows that

$$\vec{F} = -\vec{\nabla}U$$

Given that the total force acting on a particle is independent of the path, it is easy to show that it is conservative. That is, it allows for a conserved energy. To see this, let us compute the work done by the force from some point $\vec{r}_1$ to $\vec{r}_2$. Then, using the Work-Energy Theorem, we get that

$$\begin{aligned} K_2 - K_1 &= \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \\ &= -\int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla} U \cdot d\vec{r} \\ &= -\int_{\vec{r}_1}^{\vec{r}_2} dU \\ &= -U(\vec{r}_2) + U(\vec{r}_1) \end{aligned}$$

which gives

$$K_2 + U(\vec{r}_2) = K_1 + U(\vec{r}_1)$$

This tells us that the conserved energy is

$$E = K + U(\vec{r})$$

Potential Energy

We have seen that if a force is conservative, there exists a 'Potential Energy' function U such that

$$\vec{F} = -\vec{\nabla} U$$

It is however clear that this function cannot be unique, since it we add an arbitrary constant to it, this equation will still be satisfied. Equivalently, the equation

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} d\vec{r} \cdot \vec{F}$$

makes no reference to the 'reference' point $\vec{r}_0$ and the arbitrariness in defining U could be equivalently seen to stem from this. This arbitrariness is not of physical relevance. What is of physical relevance is that the sum of kinetic and potential energy is conserved. If we add a constant to a conserved quantity, the resulting quantity will still be conserved. However, *changes* in potential energy are of physical relevance, since there is no arbitrariness there (the arbitrary constant cancelling when we take the difference in potential energy between two points). It is clear from the definition of U that the change in poential energy from point $\vec{r}_1$ to $\vec{r}_2$ is

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \cdot \vec{F}$$

which is not arbitrary.

Example

Constant Force Let a constant force $\vec{F}$ act on a particle. This force does satisfy $\vec{\nabla} \times \vec{F} = 0$, so is conservative. Choosing any point $\vec{r}_0$ as a reference, the potential energy of the particle at point $\vec{r}$ is given by

$$\begin{aligned} U(\vec{r}) &= -\int_{\vec{r}_0}^{\vec{r}} d\vec{r} \cdot \vec{F} \\ &= -\vec{F} \cdot \left(\int_{\vec{r}_0}^{\vec{r}} d\vec{r} \right) \\ &= -\vec{F} \cdot (\vec{r} - \vec{r}_0) \end{aligned}$$

Then apart from an additive constant, a possible potential energy function is

$$U(\vec{r}) = -\vec{F} \cdot \vec{r}$$

Example

Central Force A central force $\vec{F} = f(r)\hat{r}$ has zero curl. The potential energy function will be given by

$$U(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} f(r)\,\hat{r}\cdot d\vec{r}$$

Expressing the line element $d\vec{r}$ in spherical polar coordinates, we have

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\,\sin\theta\,d\phi\,\hat{\phi}$$

Then $\hat{r}\cdot d\vec{r} = dr$, so that

$$U(\vec{r}) = -\int_{r_0}^{r} dr\,f(r)$$

which is independent of the path, and depends only on the radial distance of point $\vec{r}$ from the force centre. For instance, for an inverse square force (Gravitational, Coulomb) of the form $f(r) = c/r^2$ where c is a constant, we get

$$\begin{aligned} U(\vec{r}) &= -c\int_{r_0}^{r} dr\,\frac{1}{r^2} \\ &= c\left(\frac{1}{r} - \frac{1}{r_0}\right) \end{aligned}$$

so that a possible potential energy function is $U(r) = c/r$.

Let us now consider a complicated system of N particles, all interacting with central forces of the kind $\vec{F}_{ij} = f_{ij}(r_{ij})\hat{r}_{ij}$. Let us consider a subsystem of this system which contains n particles. Each of these particles experiences forces due to other particles of the subsystem and that due to particles belonging to the *external* environment (which is just the rest of the $N - n$ particles. If $\vec{F}_i$ is the total force acting on the i^{th} particle, we can write it as

$$\vec{F}_i = \vec{F}_i^{int} + \vec{F}_i^{ext}$$

where $\vec{F}_i^{int} = \sum_{j \in sub} \vec{F}_{ji}^{int}$ is the total force exerted by other particles of the subsystem and $\vec{F}_i^{ext}$ is the total force acting on the particle due to the environment. under the influence of each other and the environment, let the subsystem evolve from some initial state to another state. Then, it follows from the Work-energy theorem that

$$\int_{in}^{fin} \vec{F}_i \cdot d\vec{r}_i = K_i^{fin} - K_i^{in}$$

where K_i is the kinetic energy of the i^{th} particle. Therefore

$$\Delta K_{sub} = \sum_{i \in sub} \int_{in}^{fin} \vec{F}_i \cdot d\vec{r}_i$$

where K_{sub} is the total kinetic energy of the subsystem. Let W_{ext} be the work done on the subsystem by the environment.

Then

$$
\begin{aligned}
\Delta K_{sub} &= \sum_{i \in sub} \int_{in}^{fin} \vec{F}_i \cdot d\vec{r}_i \\
&= \sum_{i \in sub} \int_{in}^{fin} \vec{F}_i^{ext} \cdot d\vec{r}_i + \sum_{i \in sub} \int_{in}^{fin} \vec{F}_i^{int} \cdot d\vec{r}_i \\
&= W_{ext} + \sum_{j \in sub} \sum_{i \in sub} \int_{in}^{fin} \cdot d\vec{r}_i \\
&= W_{ext} + \sum_{pairs \in sub} \left(\int_{in}^{fin} \vec{F}_{ji}^{int} \cdot d\vec{r}_i + \int_{in}^{fin} \vec{F}_{ij}^{int} \cdot d\vec{r}_j \right) \\
&= W_{ext} + \sum_{pairs \in sub} \left(\int_{t_i}^{t_f} \vec{F}_{ji}^{int} \cdot \vec{v}_i \, dt + \int_{t_i}^{t_f} \vec{F}_{ij}^{int} \cdot \vec{v}_j \, dt \right) \\
&= W_{ext} + \sum_{pairs \in sub} \int_{t_i}^{t_f} dt \left(\vec{F}_{ji}^{int} \cdot \vec{v}_i + \vec{F}_{ij}^{int} \cdot \vec{v}_j \right) \\
&= W_{ext} + \sum_{pairs \in sub} \int_{t_i}^{t_f} dt \, \vec{F}_{ji}^{int} \cdot (\vec{v}_i - \vec{v}_j) \\
&= W_{ext} - \sum_{pairs \in sub} \int_{t_i}^{t_f} dt \, \vec{F}_{ji}^{int} \cdot \frac{d\vec{r}_{ji}}{dt}
\end{aligned}
$$

Then the work done by the environment on the subsystem is

$$\begin{aligned} W_{ext} &= \Delta K_{sub} + \sum_{pairs \in sub} \int_{t_i}^{t_f} dt\ \vec{F}_{ji}^{int} \cdot \frac{d\vec{r}_{ji}}{dt} \\ &= \Delta K_{sub} + \sum_{pairs \in sub} \int_{t_i}^{t_f} dt\ f_{ji}(r_{ji}) \hat{r}_{ji} \cdot \frac{d\vec{r}_{ji}}{dt} \\ &= \Delta K_{sub} + \sum_{pairs \in sub} \int_{t_i}^{t_f} dt\ f_{ji}(r_{ji}) \frac{dr_{ji}}{dt} \end{aligned}$$

Writing

$$f_{ij}(r_{ij}) = \frac{dU_{ij}(r_{ij})}{dr_i j}$$

we get

$$\begin{aligned} W_{ext} &= \Delta K_{sub} + \sum_{pairs \in sub} \int_{t_i}^{t_f} dt\ \frac{dU_{ji}(r_{ji})}{dr_{ji}} \frac{dr_{ji}}{dt} \\ &= \Delta K_{sub} + \sum_{pairs \in sub} \int_{i}^{f} \frac{dU_{ji}(r_{ji})}{dr_{ji}} dr_{ji} \\ &= \Delta K_{sub} + \sum_{pairs \in sub} \int_{i}^{f} dU_{ji} \\ &= \Delta K_{sub} + \Delta U_{internal} \end{aligned}$$

where we define

$$U_{internal} = \sum_{pairs \in sub} U_{ij}$$

as the *internal potential energy* of the subsystem. If we define

$$E_{sub} = K_{sub} + U_{internal}$$

as the energy (not conserved) of the subsystem, then we have the following result:

Work done on a Subsystem

The work done on a subsystem by its environment equals the change in energy of the subsystem.

Note that in this expression for energy of the subsystem, we have *not* included the potential energy of interaction with the environment.

Classical Mechanics

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Outline

Illustration: Two Particles

Consider a system of two particles (of masses m_1 and m_2) interacting with a force that is *central*, that is, depends only on their separation and their relative displacement

$$\vec{F}_{12} = f(r)\hat{r}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative displacement of the two particles and $\hat{r}$ is a unit vector along the direction of $\vec{r}$. Assuming there are no external forces acting on the system, the total momentum of the system will be conserved and the centre of mass will move with a uniform velocity (equal to the total momentum divided by the total mass of the system). The equations of motion for the two particles will be

$$\begin{aligned} m_1 \frac{d^2\vec{r}_1}{dt^2} &= \vec{F}_{21} \\ &= -f(r)\,\hat{r} \\ m_2 \frac{d^2\vec{r}_2}{dt^2} &= \vec{F}_{12} \\ &= f(r)\,\hat{r} \end{aligned}$$

We can eliminate the position vectors $\vec{r}_1$ and $\vec{r}_2$ in favor of the position vector of the CM $\vec{R}_{cm}$ and the relative displacement $\vec{r}$, since

$$\begin{aligned} \vec{R}_{cm} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 \end{aligned}$$

This gives

$$\begin{aligned} \vec{r}_1 &= \vec{R}_{cm} + \frac{m_2}{m_1 + m_2}\vec{r} \\ \vec{r}_2 &= \vec{R}_{cm} - \frac{m_1}{m_1 + m_2}\vec{r} \end{aligned}$$

Differentiating these equations with respect to time, we get

$$\begin{aligned} \vec{v}_1 &= \vec{V}_{cm} + \frac{m_2}{m_1 + m_2}\vec{v} \\ \vec{v}_2 &= \vec{V}_{cm} - \frac{m_1}{m_1 + m_2}\vec{v} \end{aligned}$$

where $\vec{v} = \vec{v}_1 - \vec{v}_2$ is the relative velocity.

Substituting these in (either) equations for motion (and observing that $d^2\vec{R}_{cm}/dt^2 = 0$, we get

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right)\frac{d^2\vec{r}}{dt^2} = -f(r)\,\hat{r}$$

We write this as

$$\mu\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

is called the *reduced mass* of the system and

$$\vec{F} = -f(r)\,\hat{r}$$

is the internal force of interaction. This resembles the equation of motion of a *single* particle of mass μ under the influence of force $\vec{F}$.

Energy

Since the force is central, the total energy of the system will also be conserved. To get an expression for the conserved energy, take the dot product of equation of motion of m_1 with $\vec{v}_1$ and dot product of equation of motion of m_2 with $\vec{v}_2$ and add

$$\begin{aligned} m_1\vec{v}_1 \cdot \frac{d\vec{v}_1}{dt} + m_2\vec{v}_2 \cdot \frac{d\vec{v}_2}{dt} &= -f(r)\hat{r} \cdot (\vec{v}_1 - \vec{v}_2) \\ \Longrightarrow \frac{1}{2}m_1\frac{d\vec{v}_1^2}{dt} + \frac{1}{2}m_2\frac{d\vec{v}_2^2}{dt} &= -f(r)\hat{r} \cdot \frac{d\vec{r}}{dt} \\ \Longrightarrow \frac{dK}{dt} &= -f(r)\frac{dr}{dt} \end{aligned}$$

where K is the total kinetic energy of the system.

There always exists a function $U(r)$ such that

$$f(r) = \frac{dU}{dr}$$

Then

$$\begin{aligned} \frac{dK}{dt} &= -\frac{dU}{dr}\frac{dr}{dt} \\ &= -\frac{dU}{dt} \end{aligned}$$

which tells us that

$$E = \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 + U(r)$$

is conserved. The potential energy of the system is given by

$$U(r) = \int_{r_0}^{r} dr\ f(r)$$

where r_0 is arbitrary and has no physical significance.

The velocities $\vec{v}_1$ and $\vec{v}_2$ can be traded for the velocity of the CM and the relative velocity of the two particles. This substitution gives

$$E = \frac{1}{2} M \vec{V}_{cm}^2 + \frac{1}{2} \mu \vec{v}^2 + U(r)$$

where $M = m_1 + m_2$ is the total mass of the system. Since $\vec{V}_{cm}$ is itself constant, the first term (which we can call 'energy of CM' or energy of mass motion) is itself conserved. Of interest is the conservation of the second term. In particular, in the CM frame, the first term will be zero and the expression for conserved energy becomes

$$E = \frac{1}{2} \mu \, \vec{v}^2 + U(r)$$

We can also obtain this expression directly from the effective one particle equation of motion

$$\mu \frac{d^2 \vec{r}}{dt^2} = -f(r) \, \hat{r}$$

The effective force if $\vec{F} = -f(r) \hat{r}$ and is such that $\vec{\nabla} \times \vec{F} = 0$. Then, there exists a function $U(r)$ such that $\vec{F} = -\vec{\nabla} U$, from which we again get

$$f(r) = \frac{dU}{dr}$$

and a conserved energy

$$E = \frac{1}{2} \mu \, \vec{v}^2 + U(r)$$

Angular Momentum

For an isolated system of particles interacting through central interactions, in addition to momentum and energy, there is an additional conserved quantity, the *Angular Momentum* of the system. This is defined to be

$$\begin{aligned} \vec{L} &= \sum_i \vec{r}_i \times \vec{p}_i \\ &= \sum_i \vec{r}_i \times m_i \vec{v}_i \end{aligned}$$

Differentiating this with respect to time

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \sum_i \frac{\vec{r}_i}{dt} \times \vec{p}_i + \sum_i \vec{r}_i \times \frac{\vec{p}_i}{dt} \\ &= \sum_i \vec{v}_i \times \vec{p}_i + \sum_i \vec{r}_i \times \sum_{j \neq i} \vec{F}_{ji} \\ &= \sum_{pairs\ i,j} \left(\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} \right) \\ &= \sum_{pairs\ i,j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \\ &= 0 \end{aligned}$$

since $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ is parallel to $\vec{F}_{ij}$ (central interactions).

For a system of two particles, the angular momentum of the system has a simple form. Startign with

$$\vec{L} = m_1 \vec{r}_1 \times \vec{v}_1 + m_2 \vec{r}_2 \times \vec{v}_2$$

and substituting for $\vec{r}_1, \vec{r}_2, \vec{v}_1$ and $\vec{v}_2$ in therms of CM and relative position and velocity, we get

$$\vec{L} = M\, \vec{R}_{cm} \times \vec{V}_{cm} + \mu \vec{r} \times \vec{v}$$

Again, the first term associated with CM motion (or mass motion) of the system is itself conserved. Then, by moving to the CM frame, we can only focus on the more physically interesting piece

$$\vec{L} = \mu\, \vec{r} \times \vec{v}$$

Since $\vec{L}$ is conserved, both its magnitude and direction will be constant. In particular, it is easy to see that the motion will be confined to the plane containing $\vec{r}$ and $\vec{v}$ since $\vec{L}$ is orthogonal to both.

Summary:

Summary of Central Force Dynamics

Equation of Motion:

$$\mu \frac{d^2\vec{r}}{dt^2} = -f(r)\,\hat{r}$$

Conserved quantities:
Energy

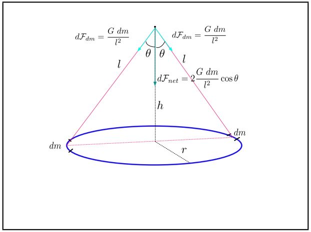
$$E = \frac{1}{2}\mu\,\vec{v}^2 + U(r)$$

Angular Momentum

$$\vec{L} = \mu\,\vec{r} \times \vec{v}$$

Gravitational Field due to a Ring

Gravitational Field: Force per unit 'test mass' (mass on which force is being exerted). We calculate gravitational field due to a massive ring at a point along the axis. We use Principle of Superposition and symmetry.

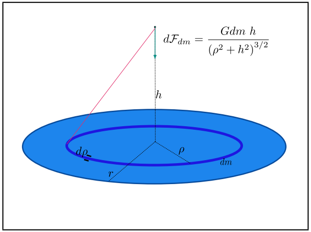


The net gravitational field will be along the axial direction since pairs of corresponding masses *dm* will contribute together to give a net contribution along the axis. The vertical component due to any mass element *dm* will be $(Gdm\cos\theta/l^2)$. Since θ and l are the same for each element, the total gravitational field will have magnitude

$$\mathcal{F} = \frac{Gm}{l^2}\cos\theta = \frac{Gm\,h}{\left(r^2+h^2\right)^{3/2}}$$

Gravitational Field due to a Disc

We compute the gravitational field due to a disc of radius r and mass m at a point along its axis.



We subdivide the Disc into concentri rings of radii ρ and radial thickness $d\rho$. The area of each such ring will be $dV = 2\pi\rho$. The mass of the ring will be

$$\begin{aligned} dm &= m\frac{dV}{\pi r^2} \\ &= \frac{2m\ \rho d\rho}{r^2} \end{aligned}$$

The gravitational filed due to such a ring will be

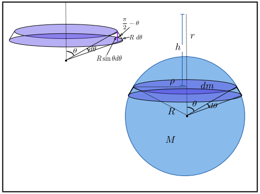
$$\begin{aligned} d\mathcal{F} &= \frac{Gdm\ h}{\left(\rho^2 + h^2\right)^{3/2}} \\ &= \frac{2mGh}{r^2} \frac{\rho d\rho}{\left(\rho^2 + h^2\right)^{3/2}} \end{aligned}$$

The total gravitational field due to the disc will be

$$\begin{aligned} \mathcal{F} &= \frac{2mGh}{r^2} \int_0^r d\rho \frac{\rho}{\left(\rho^2 + h^2\right)^{3/2}} \\ &= \frac{2mG}{r^2} \left(1 - \frac{h}{\sqrt{r^2 + h^2}}\right) \end{aligned}$$

Gravitational Field due to a Sphere

We divide the sphere into discs and use superposition to calculate the net gravitational field.



Gravitational field due to the dic is

$$d\mathcal{F} = \frac{2dmG}{r^2}\left(1 - \frac{h}{\sqrt{\rho^2 + h^2}}\right)$$

The mass of the disc is

$$dm = M\left(\frac{dV}{\frac{4}{3}\pi R^3}\right)$$

where dV is the volume of the disc, equal to the area of the circular surface $\pi\rho^2$ times its height $R\sin\theta d\theta$.

Then

$$d\mathcal{F} = \frac{3GM}{2R^2} \sin\theta d\theta \left(1 - \frac{h}{\sqrt{\rho^2 + h^2}}\right)$$

Using $\rho = R\sin\theta$ and $h = r - R\cos\theta$, the total gravitational field is

$$\mathcal{F} = \frac{3GM}{2R^2} \int_0^{\pi} d\theta \sin\theta \left(1 - \frac{r - R\cos\theta}{\sqrt{r^2 + R^2 - 2rR\cos\theta}}\right)$$

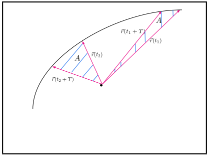
The integral is easy to perform and yields

$$\mathcal{F} = \frac{GM}{r^2}$$

which is the same as if all the mass of the sphere were concentrated at a single point at its centre.

Consequences of Energy and Angular Momentum Conservation

Consequences of Angular Momentum conservation: We have already seen that as a consequence of angular momentum conservation, the motion of two objects under a central force is confined to a plane. A further consequence is as follows: as the objects move in a plane, the length and orientation of the relative displacement vector changes with time. However, this happens such that the relative displacement vector sweeps equal areas in equal intervals of time



Proof: We choose polar coordinates r, θ. Then if the radius vector sweeps an infinitesimal angle $d\theta$ in an infinitesimal interval dt, then the area swept will be

$$dA = \frac{1}{2} r^2 d\theta$$

The rate of change of this area is then

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

In the CM frame, the angular momentum of the system is $\vec{L} = \mu \vec{r} \times \vec{v}$. In polar coordinates, $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$. Then the magnitude of angular momentum is

$$L = \mu r^2 \dot{\theta}$$

which is constant. Then,

$$\frac{dA}{dt} = \frac{L}{2\mu}$$

will be constant. Then, in a given interval of time, the area swept will be the same. The conservation of angular momentum allows us to eliminate $\dot{\theta}$ in favor of r

$$\dot{\theta} = \frac{L}{\mu r^2}$$

This makes the analysis of the dynamics very simple.

Consequences of Conservation of Energy: In the CM frame, the energy of the system is

$$E = \frac{1}{2}\mu\vec{v}^2 + U(r)$$

Expressing velocity in polar coordinates, and trading $\dot{\theta}$ for r, we get

$$\begin{aligned} E &= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + U(r) \\ &= \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) \end{aligned}$$

where

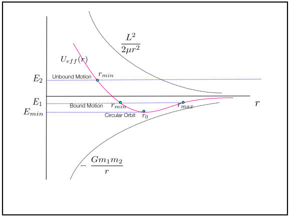
$$U_{eff}(r) = U(r) + \frac{L^2}{2\mu r^2}$$

is called the *Effective Potential*. Note that the real potential energy function is $U(r)$. However, the conservation of angular momentum gives rise to an expression for anergy exclusively in terms of the radial coordinate and its derivative. Then, so far as the radial motion is concerned, the expression for energy is as if there is a particle moving in one dimension under the influence of a 'potential energy' function $U_{eff}(r)$. The angular momentum dependent term is called a *centrifugal barrier* term. If this were the only term, it would give rise to a 'repulsive' radial force.

Given the form of the physical potential energy function, it is possible, just by analysing the behavior of $U_{eff}(r)$ as a function of r, to comment on the general motion of the two particle system. In particlular, it gives an insight into whether the particles can be bound in a confined region of space in an orbit. As an example, consider the gravitational interaction between two masses, such that

$$U_{eff}(r) = -\frac{Gm_1 m_2}{r} + \frac{L^2}{2\mu r^2}$$

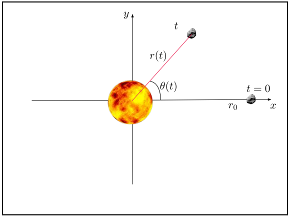
Depending on the (conserved) energy of the system, the system may or may not be confined to an orbit.



The core idea is that the 'kinetic energy' term $\mu\dot{r}^2/2$ cannot be negative. Points at which $\dot{r} = 0$ are such that the radial distance (momentarily) does not change. At these points, the particles are moving perpendicular to the line joining them. These are called 'turning points', since at these points, the sign of $\dot{r}$ changes. there will always be at least one such turning point. This will be points at which the line of constant energy intersects the effective potential energy function (plotted as a function of r). For a special (minimum) value of energy, this line will graze the minimum of the effective potential. For this energy, r will not change, and the particles will execute circular motion. If the energy is incresed, within a certain range, there will be two turning points. For these values of energy, the separation between the two particles will always lie between these points and they will form a bound motion (for the gravbitational interaction, this corresponds to an elliptical orbit). for large enough values of energy, there will be only one turning point, and this will be the minimum distance between the two particles. The maximum distance is not bounded. For gravitational interaction, this corresponds to parabolic and hyperbolic orbits.

Celestial Orbits

We now turn to gravitational interaction of celestial objects with the Sun in the Solar system. Since the mass of the Sun is much larger than any other celestial object in the Solar system, we ignore its dynamics and assume the CM to lie at its centre. Then $\mu \sim m$, the mass of the celestial object. The goal is to construct the detailed orbit of any celestial object. say, we are given the position and velocity of the object at any instant of time (say $t = 0$). We can choose a polar coordinate system with the polar angle θ measured relative to the line joining the centre of the sun to the position of the object at this instant. Then, information about position and velocity is equivalent to information about $r, \dot{r}, \dot{\theta}$ at $t = 0$. Of course, $\theta = 0$ at this instant by construction.



To determine the radial motion, we use the expression for the conserved energy, which gives the following expression for $\dot{r}$

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{m}\left(E - U_{eff}\right)}$$

The sign is chosen depending on whether the radial distance is increasing or decreasing during the period over which change in r is to be determined. Assuming the positive sign, this equation allows us to explicitly compute the radial coordinate at instant t, given the coordinate r_0 at an instant t_0

$$\int_{r_0}^{r} \frac{dr}{\sqrt{(2/m)\left(E - U_{eff}\right)}} = t - t_0$$

The angular coordinate can be determined from the expression for the conserved angular momentum

$$\begin{aligned} \frac{d\theta}{dt} &= \frac{L}{mr^2} \\ \Longrightarrow \theta - \theta_0 &= \frac{L}{m}\int_{t_0}^{t} \frac{dt}{r^2} \end{aligned}$$

where in the t integral, it is assumed we know the function $r(t)$ already, which is determined from the energy relation.

The Orbit

The shape of the orbit is determined if we know r as a function of θ. Taking the ratio of $\dot{\theta}$ and $\dot{r}$, we get

$$\begin{aligned} \frac{d\theta}{dr} &= \frac{L}{mr^2}\sqrt{\frac{m}{2(E-U_{eff})}} \\ \implies \theta - \theta_0 &= L\int_{r_0}^{r} \frac{dr}{r^2\sqrt{2m(E-U_{eff})}} \end{aligned}$$

For motion of celestial objects

$$U = -\frac{C}{r}$$

where $C = GMm$ where M is the mass of the Sun. However, the same result would be valid for motion of an object under the gravitational influence of a much more massive object (satellite around the Earth,etc.).

Then

$$\theta - \theta_0 = L \int_{r_0}^{r} \frac{dr}{r\sqrt{2m\left(2mEr^2 + 2mCr - L^2\right)}}$$

The integral yields the solution

$$r = \frac{\left(L^2/mC\right)}{1 - \sqrt{1 + \left(2EL^2/mC^2\right)}\ \sin(\theta - \theta_0)}$$

The effective potential energy function

$$U_{eff}(r) = -\frac{C}{r} + \frac{L^2}{2mr^2}$$

has a minimum at $r_0 = L^2/mC$ and this minimum value is $U_{eff}^{min} = -mC^2/2L^2$. The constant θ_0 is arbitrary, and setting it $\theta_0 = -\pi/2$, we get the following equation for orbit

$$r = \frac{r_0}{1 - \epsilon \cos\theta}$$

where

$$\epsilon = \sqrt{1 + \frac{E}{\left|U_{eff}^{min}\right|}}$$

is a dimensionless parameter called *eccentricity*.

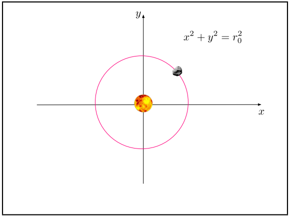
Clearly, since the least value of energy is equal to U_{eff}^{min}, therefore $0 \leq \epsilon < \infty$. We first express the equation of the orbit in Cartesian coordinates. Using $x = r\cos\theta$ and $r = \sqrt{x^2 + y^2}$, we get

$$(1-\epsilon^2)x^2 - 2r_0\epsilon x + y^2 = r_0^2$$

For the lowest value of energy, $\epsilon = 0$ and the equation reduces to

$$x^2 + y^2 = r_0^2$$

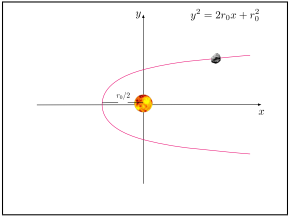
which is the equation for a circular orbit. Then, for the least value of energy, the celestial object moves in a circular orbit



For $\epsilon = 1$, the equation becomes

$$y^2 = 2r_0x + r_0^2$$

which is the equation for a parabola



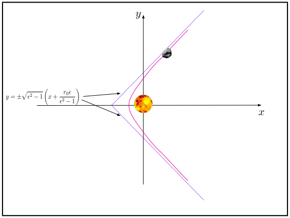
For $\epsilon > 1$, we write the equation in the form

$$\begin{aligned} y^2 &= (\epsilon^2 - 1)x^2 + 2r_0\epsilon x + r_0^2 \\ &= (\epsilon^2 - 1)\left(x + \frac{r_0\epsilon}{\epsilon^2 - 1}\right)^2 - \frac{r_0^2}{\epsilon^2 - 1} \end{aligned}$$

This is the equation for a hyperbola. The hyperbola is bounded within two lines

$$y = \pm\sqrt{\epsilon^2 - 1}\left(x + \frac{r_0\epsilon}{\epsilon^2 - 1}\right)$$

These are *asymptotes* to the hyperbola



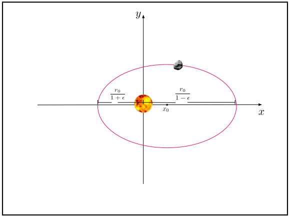
Finally, for $0 < \epsilon < 1$, the equation can be written as

$$(1-\epsilon^2)\left(x - \frac{r_0\epsilon}{1-\epsilon^2}\right)^2 + y^2 = \frac{r_0^2}{1-\epsilon^2}$$

which is of the form

$$a(x-x_0)^2 + y^2 = R^2$$

which is the equation of an ellipse with $a = 1-\epsilon^2$, $x_0 = r_0\epsilon/(1-\epsilon^2)$ and $R = r_0/\sqrt{1-\epsilon^2}$. Clearly, $a < 1$



The centre of the ellips is *not* at the location of the massive object. The massive object is instead at the *focus* of the eelipse. From the equation $r = r_0/1 - \epsilon\cos\theta$, it is clear that the orbiting object will be closest for $\theta = \pi$, for which the distance is $r_p = r_0/1+\epsilon$ (*Perihelion* distance) and farthest for $\theta = 0$, for which the distance is $r_p = r_0/1-\epsilon$ (*Aphelion* distance)

Period of Elliptic Orbit: To calculate the tiume period of the elliptic orbit, we move to the angular momentum equation $L = mr^2\dot{\theta}$ and integrate over one time period (for which the angle changes over 2π

$$\begin{aligned} T &= \frac{m}{L}\int_0^{2\pi} r^2 d\theta \\ &= \frac{mr_0^2}{L}\int_0^{2\pi} \frac{d\theta}{(1-\epsilon\cos\theta)^2} \end{aligned}$$

where the equation of the orbit has been used. The integral over θ is standard and gives

$$\int_0^{2\pi} \frac{d\theta}{(1-\epsilon\cos\theta)^2} = \frac{2\pi}{\left(1-\epsilon^2\right)^{3/2}}$$

Then,

$$T = \frac{2m\pi r_0^2}{L\left(1-\epsilon^2\right)^{3/2}}$$

Squaring both sides and using the fact that the length of the major axis is $A = 2r_0/(1-\epsilon^2)$ and that $r_0 = L^2/mC$, we get

$$T^2 = \frac{m\pi^2}{2C} A^3$$

Classical Mechanics

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Outline

Angular Momentum

Consider a system of particles. We have seen that if the particles interact through central forces then in addition to energy and mmentum, there exists another physical quantity, Angular Momentum, which is conserved. Relative to an origin, the angular momentum of a particle of mass m is defined as

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ &= m\,\vec{r} \times \vec{v} \end{aligned}$$

In absence of a force, this does not change with time. If an external force $\vec{F}$ acts on the particle then the rate of change of angular momentum is given by

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \vec{r} \times \frac{d\vec{p}}{dt} \\ &= \vec{r} \times \vec{F} \end{aligned}$$

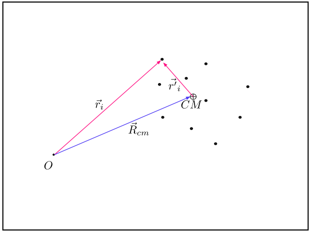
The quantity $\vec{r} \times \vec{F}$ is called *torque*. Then

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

For a system of particles experiencing both (central) internal and external forces, the total angular momentum $\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$ changes according to

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} \\ &= \sum_i \vec{r}_i \times \left(\vec{F}_i^{ext} + \sum_{j\neq i} \vec{F}_{ji}\right) \\ &= \sum_i \vec{r}_i \times \vec{F}_i^{ext} \\ &= \vec{\tau}_{ext}\end{aligned}$$

where the internal contributions cancel (as demonstrated before). The quantity $\vec{\tau}_{ext} = \sum_i \vec{r}_i \times \vec{F}_i^{ext}$ is the total external torque on the system of particles.



We can use the CM of the system as an origin as well. If $\vec{r'}_i$ is the position of the i^{th} particles relative to the CM and $\vec{r}_i$ its position relative to the chosen origin O, then

$$\vec{r}_i = \vec{R}_{cm} + \vec{r'}_i$$

Similar relation holds for velocity relative to O and the CM

$$\vec{v}_i = \vec{V}_{cm} + \vec{v'}_i$$

Then the angular momentum of the system relative to O can be written as

$$\begin{aligned}
\vec{L} &= \sum_i m_i \left(\vec{R}_{cm} + \vec{r'}_i\right) \times \left(\vec{V}_{cm} + \vec{v'}_i\right) \\
&= \sum_i m_i \, \vec{R}_{cm} \times \vec{V}_{cm} + \sum_i m_i \, \vec{r'}_i \times \vec{v'}_i + \vec{R}_{cm} \times \left(\sum_i m_i \, \vec{v'}_i\right) + \left(\sum_i m_i \, \vec{r'}_i\right) \times \vec{V}_{cm} \\
&= M\vec{R}_{cm} \times \vec{V}_{cm} + \sum_i m_i \, \vec{r'}_i \times \vec{v'}_i \\
&= \vec{L}_{cm} + \vec{L}'
\end{aligned}$$

where $\vec{L}_{cm} = M\vec{R}_{cm} \times \vec{V}_{cm}$ is termed ‘angular momentum of CM’ and $\vec{L}' = \sum_i m_i \, \vec{r'}_i \times \vec{v'}_i$ is the angular momentum of the system about the CM. The other two terms are identically zero, since $\sum_i m_i \, \vec{v'}_i$ is simply the momentum of the system as viewed in the CM frame, which is zero and $\sum_i m_i \, \vec{r'}_i$ equals the mass of the system times the position of the CM relative to itself, which is zero as well.

Then the rate of change of angular momentum about O is given by

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_{cm}}{dt} + \frac{d\vec{L}'}{dt}$$

where

$$\begin{aligned} \frac{d\vec{L}_{cm}}{dt} &= M\frac{d\vec{R}_{cm}}{dt} \times \vec{V}_{cm} + M\vec{R}_{cm} \times \frac{d\vec{V}_{cm}}{dt} \\ &= \vec{R}_{cm} \times M\,\frac{d\vec{V}_{cm}}{dt} \\ &= \vec{R}_{cm} \times \vec{F}_{ext} \end{aligned}$$

where $\vec{F}_{ext}$ is the total external force acting on the system. Therefore

$$\frac{d\vec{L}}{dt} = \vec{R}_{cm} \times \vec{F}_{ext} + \frac{d\vec{L}'}{dt}$$

We now look at the expression for the total torque on the system due to external forces. Once again, expressing $\vec{r}_i = \vec{R}_{cm} + \vec{r'}_i$, we get

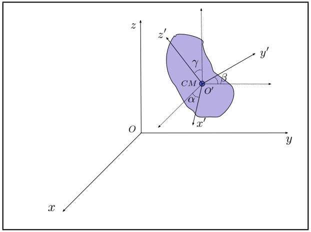
$$
\begin{aligned}
\vec{\tau}_{ext} &= \sum_i \vec{r}_i \times \vec{F}_i^{ext} \\
&= \sum_i \left(\vec{r'}_i + \vec{R}_{cm} \right) \times \vec{F}_i^{ext} \\
&= \sum_i \vec{r'}_i \times \vec{F}_i^{ext} + \vec{R}_{cm} \times \sum_i \vec{F}_i^{ext} \\
&= \vec{R}_{cm} \times \vec{F}_{ext} + \sum_i \vec{r'}_i \times \vec{F}_i^{ext} \\
&= \vec{\tau}_{cm} + \vec{\tau'}
\end{aligned}
$$

where $\vec{\tau}_{cm} = \vec{R}_{cm} \times \vec{F}_{ext}$ can be formally identified as the 'torque acting on the CM' and $\vec{\tau'} = \sum_i \vec{r'}_i \times \vec{F}_i^{ext}$ is the torque acting on the system about the CM. Then, it follows that

$$
\begin{aligned}
\frac{d\vec{L}}{dt} &= \vec{R}_{cm} \times \vec{F}_{ext} + \frac{d\vec{L}'}{dt} \\
&= \vec{\tau}_{cm} + \vec{\tau'}
\end{aligned}
$$

Motion of a Rigid Body

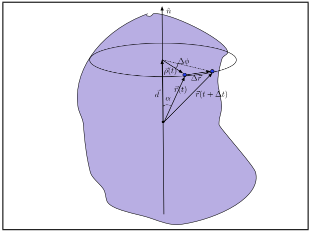
In a ‘rigid’ object, the relative positions of the constituent particles remain fixed. Then, if there are N particles in the object, to describe the motion of the object, we do not need to describe how $3N$ coordinates (three for each particle) change with time. Instead, the motion is completely described by describing how six coordinates change with time. It is convenient to view the motion as that of the CM and the motion about the CM. We need three coordinates to describe the motion of the CM. Then, relative to the CM, since the position of the CM itself does not change and also the relative positions of the constituent particles do not change, the only motion about the CM is purely rotational. To see this, let us set up a coordinate system with origin O and another coordinate system with the origin located at the CM such that the coordinate aces are rigidly attached to the object



The x', y', z' coordinates of all the particles will be fixed, since these coordinates are attached to the object. Then, to describe the coordinates of any particle with respect to the coordinate system x, y, z, all we need is to determine is the position of the CM (the other origin) and the orientation of system x', y', z' relative to system x, y, z (given in terms of three angles between the respective coordinate axes). Therefore, we just need six coordinates to specify the entire configuration of the rigid object. An equivalent point of view is that since the motion about the CM is a pure rotation, it must be about an axis, and by some angle. The direction of the axis can be described by two numbers (say polar angles θ and ϕ relative to the coordinate system x, y, z) and the angle of rotation about that axis gives a third number. Again, the motion about the CM is described in terms of three numbers. For simplicity, we will consider only those situations in which the direction of the axis of rotation does not change with time. Then the entire motion of the rigid object can be described by the position of the CM and a single angle of rotation about it. For instance, a cylinder rolling down an incline is an example in which the direction of the axis of rotation is fixed.

Fixed Axis Rotation

Consider a rigid body rotating about a fixed axis. For this discussion, we are not assuming that the axis is passing through the CM. As the body rotates, every particle moves in a circle with radius equal to the perpendicular distance from the particle to the axis. Given an initial configuration of the object, the new configuration when the object rotates by angle ϕ is simple to describe: any one particle has moved along its circular path by angle ϕ, the same for all the particles. We can define an angular coordinate ϕ such that it is zero for any one instant of time and at a later instant of time is equal to the angle through which the body rotates in that duration. To predict the motion of the object from the dynamical equation, we also need to know the velocity of every particle at a given instant of time, apart from its position. We now show that the velocity of every particle is known, if we know the *angular velocity* of the rigid body.



Let the position of the particle at instant t relative to an origin on the axis be $\vec{r}(t)$. As the body turns through an infinitesimal angle $\Delta\phi$ in time interval Δt, its new position is $\vec{r}(t+\Delta t)$. Let the perpendicular distance of the particle from the axis be ρ. From the illustration, we see that

$$\vec{r}(t) = \vec{d} + \vec{\rho}(t)$$

where $\vec{d}$ is a vector along the axis, and the vector $\vec{\rho}(t)$ changes with time because its direction changes with time (its tip traces a circle along with the particle). Further,

$$\vec{r}(t+\Delta t) = \vec{r}(t) + \Delta\vec{r}$$

where in the limit Δt is infinitesimal, $\left|\Delta\vec{r}\right| = \rho\,\Delta\phi$. Then it is clear that

$$\Delta\vec{r} \quad = \quad \hat{n} \times \vec{\rho}(t)\,\Delta\phi$$

However, since $\hat{n} \times \vec{d} = 0$, it follows that

$$\Delta\vec{r} = \hat{n} \times \vec{r}(t)\ \Delta\phi$$

Dividing by Δt and taking the limit $\Delta t \to 0$, we get

$$\frac{d\vec{r}}{dt} = \omega\hat{n} \times \vec{r}(t)$$

where $\omega = \dot{\phi}$ is the *angular speed*. We define the instantaneous *angular velocity* as $\vec{\omega} = \dot{\phi}\hat{n}$. Then, we get

$$\vec{v}(t) = \vec{\omega}(t) \times \vec{r}(t)$$

It is clear that if the vector $\vec{\omega}$ is known (the same for all particles), we can determine the velocity of any particle of the object.

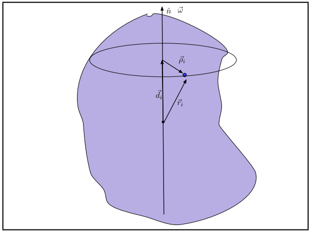
Rigid Body Dynamics

The dynamical problem is as follows: Given the position and velocity of all the particles at any one instant of time, we need to determine them at a later instant. If we know the positions at any one instant of time and the angular velocity vector at all instants of time, we can in principle solve the differential equation

$$\frac{d\vec{r}}{dt} = \vec{\omega}(t) \times \vec{r}(t)$$

to determine the position and velocity at all instants of time. This is just a first order differential equation in $\vec{r}(t)$, provided that $\vec{\omega}(t)$ is a known function of time. The problem then reduces to the problem of determining a dynamical equation for the angular velocity. It is here that angular momentum and torque take centrestage.

Angular Momentum of a Rigid Body



Let us compute the angular momentum of a rigid body spinning about a fixed axis with angular velocity $\vec{\omega}$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

Using the fact that $\vec{v}_i = \vec{\omega} \times \vec{r}_i$ (and referring to the illustration)

$$
\begin{aligned}
\vec{L} &= \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \\
&= \sum_i m_i \left(\vec{d}_i + \vec{\rho}_i\right) \times (\vec{\omega} \times \vec{\rho}_i) \\
&= \sum_i m_i \vec{\rho}_i \times (\vec{\omega} \times \vec{\rho}_i) + \sum_i m_i \vec{d}_i \times (\vec{\omega} \times \vec{\rho}_i) \\
&= \vec{L}_\omega + \sum_i m_i \vec{d}_i \times (\vec{\omega} \times \vec{\rho}_i)
\end{aligned}
$$

where

$$
\vec{L}_\omega = \sum_i m_i \vec{\rho}_i \times (\vec{\omega} \times \vec{\rho}_i)
$$

From the geometry of the cross products, it is easy to see that in the expression for vector $\vec{L}_\omega$, each term is in the same direction as $\vec{\omega}$. Further, the magnitude of this vector is $L_\omega = \left(\sum_i m_i \rho_i^2\right) \omega$.

Then, we can write

$$\vec{L}_\omega = I\,\vec{\omega}$$

where

$$I = \sum_i m_i \rho_i^2$$

is called the *Moment of Inertia* about the given axis. The second term

$$\vec{L}_\perp = \sum_i m_i \vec{d}_i \times (\vec{\omega} \times \vec{\rho}_i)$$

is easily seen to be perpendicular to $\vec{\omega}$ and is not zero in general. Setting up coordinates such that the axis of rotation coincides with the z axis, we can write $\vec{\omega} = \omega\hat{k}$, $\vec{\rho}_i = x_i\hat{\imath} + y_i\hat{\jmath}$ and $\vec{d}_i = z_i\hat{k}$. Then,

$$\begin{aligned}\vec{L}_\perp &= \omega \sum_i m_i z_i \hat{k} \times \left(\hat{k} \times \left(x_i\hat{\imath} + y_i\hat{\jmath}\right)\right) \\ &= -\omega \left(\sum_i m_i z_i x_i\right)\hat{\imath} + \omega \left(\sum_i m_i z_i y_i\right)\hat{\jmath}\end{aligned}$$

In the special case the mass distribution of the body is symmetric about the axis of rotation (z axis), each sum is zero (convine yourself) and the perpendicular contribution to the total angular momentum is zero. Then, for an *axisymmetric* object,

$$\vec{L} = I\,\vec{\omega}$$

Then the fundamental dynamical quantity of interest directly related to the angular velocity is angular momentum. If we know how the angular momentum (technically, the component of angular momentum along the axis of rotation) of the body changes with time, we can predict how its angular velocity changes with time. We know the dynamical equation for $\vec{L}$

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$$

Then, we need to compute the total external torque, in particular, the component of the torque along the axis of rotation. If $\hat{n}$ is a unit vector along the axis of rotation, then it follows from the expression for the angular momentum of the rigid body that

$$\vec{L} \cdot \hat{n} = I\omega$$

Differentiating with respect to time, we get

$$\begin{aligned} \hat{n} \cdot \frac{d\vec{L}}{dt} &= I\frac{d\omega}{dt} \\ \implies \vec{\tau}_{ext} \cdot \hat{n} &= I\frac{d\omega}{dt} \end{aligned}$$

This is the rotational analogue of the equation for the CM. The quantity

$$\alpha = \frac{d\omega}{dt}$$

is called angular acceleration. The moment of inertia I is the analogue of the mass of the body.

Energy of Rotation

Let us calculate the kinetic energy of a rigid body spinning with angular velocity $\vec{\omega}$. Referring to the previous illustration, we have

$$\begin{aligned} K &= \frac{1}{2}\sum_i m_i \vec{v}_i^2 \\ &= \frac{1}{2}\sum_i m_i \left(\vec{\omega} \times \vec{\rho}_i\right)^2 \\ &= \frac{1}{2}\sum_i m_i \rho_i^2 \; \omega^2 \\ &= \frac{1}{2} I \, \omega^2 \end{aligned}$$

Work-energy Theorem for a Rigid Body

Again, confining for a moment to purely rotational motion, the generalised Work-Energy Theorem tells us that the work done by external forces on a system of particles (here the particles of a rigid body) is equal to the change in the enegy of the system, the energy beoing the sum of the kinetic energy of the particles and their mutual interaction potential energy

$$W_{ext} = \Delta E$$

where

$$E = \sum_i K_i + \sum_{pairs\ i,j} U_{ij}(r_{ij})$$

Since the separation between the particles of a rigid object does not change, there is no change in the potential energy. Then,

$$W_{ext} = \Delta K$$

where K is the rotational kinetic energy of the body.

Motion Involving Translation and Rotation

For a general motion of a rigid object involving translation and rotation, it is convenient to view the motion as motion of the CM plus rotational motion about the CM. The motion of the CM satisfies equations

$$\begin{aligned} \frac{d\vec{R}_{cm}}{dt} &= \vec{V}_{cm} \\ M\frac{d\vec{V}_{cm}}{dt} &= \vec{F}_{ext} \end{aligned}$$

The rotational motion about the CM will satisfy

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \vec{\tau}_{ext} \\ \vec{L} &= I\,\vec{\omega} + \vec{L}_{\perp} \end{aligned}$$

where $\vec{L}$ is the angular momentum about the CM and $\vec{\tau}_{ext}$ is the external torque about the CM. For an axisymmetric object, $vecL = I\,\vec{\omega}$. Often, due to constrained motion, the two motions will be related, for instance, for an object rolling without slipping.

Total Kinetic Energy

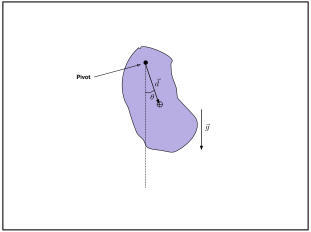
For motion involving both translations and rotations, we compute the expression for the total kinetic energy of the rigid object. Let the velocity of the i^{th} particle of the body be $\vec{v}_i$ in an inertial frame and $\vec{v'}_i$ be the velocity relative to the CM. Then the toal kinetic energy of the object is

$$\begin{aligned} K &= \frac{1}{2}\sum_i m_i \vec{v}_i^2 \\ &= \frac{1}{2}\sum_i m_i \left(\vec{V}_{cm} + \vec{v'}_i\right)^2 \\ &= \frac{1}{2}\sum_i m_i \left(\vec{V}_{cm}^2 + \vec{v'}_i^2 + 2\vec{V}_{cm}\cdot\vec{v'}_i\right) \\ &= \frac{1}{2}M\vec{V}_{cm}^2 + \frac{1}{2}\sum_i m_i \vec{v'}_i^2 + 2\vec{V}_{cm}\cdot\sum_i m_i \vec{v'}_i \\ &= \frac{1}{2}M\vec{V}_{cm}^2 + K_{rot} \end{aligned}$$

where K_{rot} is the kinetic energy (of rotation) about the CM. The term $\sum_i m_i \vec{v'}_i$ is zero, since this is the total momentum in the CM frame, which is zero.

Fixed Axis Rotation: Physical Pendulum

A rigid object is free to move under the influence of (uniform) gravity about a horizontal axis. We determine its motion.



This is an example of fixed axis rotation. The angular velocity $\vec{\omega}$ will be along the direction of the axis. Let $\hat{n}$ be a unit vector along the axis, say coming out of the plane (of the illustration). Then, $\vec{\omega} = d\theta/dt\hat{n}$ and

$$\hat{n} \cdot \frac{d\vec{L}}{dt} = \hat{n} \cdot \vec{\tau}_{ext}$$

The external force is that due to gravity. Then

$$\begin{aligned}\vec{\tau}_{ext} &= \sum_i \vec{r}_i \times m_i \vec{g} \\ &= \left(\sum_i m_i \vec{r}_i\right) \times \vec{g} \\ &= \vec{R}_{cm} \times M\vec{g}\end{aligned}$$

In particluar, we find that *torque about CM due to gravity is zero*. Then, since $\vec{L} = I\,\vec{\omega} + \vec{L}_\perp$ and $\hat{n}\cdot\vec{L}_\perp = 0$, therefore

$$\begin{aligned}\hat{n}\cdot I\,\frac{d\vec{\omega}}{dt} &= \hat{n}\cdot\left(\vec{R}_{cm}\times M\vec{g}\right) \\ \Longrightarrow\ I\,\frac{d(\vec{\omega}\cdot\hat{n})}{dt} &= \hat{n}\cdot\left(\vec{d}\times M\vec{g}\right)\end{aligned}$$

where $\vec{d}$ is along the perpendicular from the axis of rotation to the CM.

Referring to the illustration, $\vec{d} \times \vec{g} = -d\ g \sin\theta\ \hat{n}$. Then the equation describing rotation is

$$I\ \frac{d^2\theta}{dt^2} = -Mgd \sin\theta$$

For small angular displacement θ, $\sin\theta \approx \theta$, so that

$$I\ \frac{d^2\theta}{dt^2} = -Mgd\ \theta$$

which is the equation for SHM with angular frequency

$$\omega = \sqrt{\frac{Mgd}{I}}$$

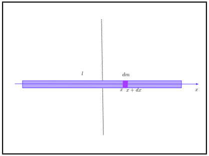
Moment of Inertia

The moment of inertia of a rigid body about its axis of rotation is given by

$$I = \sum_i m_i \rho_i^2$$

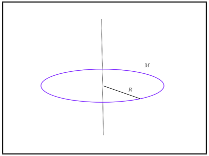
where ρ_i is the perpendicular distance of the i^{th} particle from the axis of rotation.

Moment of Inertia of a uniform rod of mass M about an axis passing through its centre



$$\begin{aligned} I &= \sum_x dm\, x^2 \\ &= \frac{M}{l} \sum_x dx\, x^2 \\ &\rightarrow \frac{M}{l} \int_{-l/2}^{l/2} dx\, x^2 \\ &= \frac{M\, l^2}{12} \end{aligned}$$

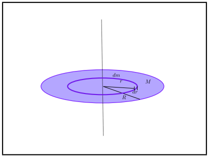
Moment of Inertia of a uniform ring of mass M and radius R about an axis passing through its centre and perpendicular to its plane



Since all the particles are at the same perpendicular distance R from the axis, the moment of inertial is simply

$$I = MR^2$$

Moment of Inertia of a uniform disk of mass *M* and radius *R* about an axis passing through its centre and perpendicular to its plane



We divide the disk into concentric rings of radii r and thickness dr. The mass of such a ring will be

$$\begin{aligned} dm &= \frac{2\pi r dr}{\pi R^2} M \\ &= \frac{2M}{R^2} r\ dr \end{aligned}$$

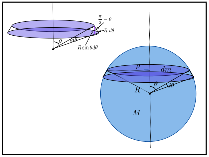
Then the moment of inertia of this ring about the axis will be

$$\begin{aligned} dI &= dm\ r^2 \\ &= \frac{2M}{R^2} r^3\ dr \end{aligned}$$

Therefore the moment of inertia of the disk will be

$$
\begin{aligned}
I &= \frac{2M}{R^2}\int_0^R dr\, r^3 \\
&= \frac{MR^2}{2}
\end{aligned}
$$

Moment of Inertia of a uniform sphere of mass *M* and radius *R* about an axis passing through its diameter



We divide the sphere into disks as illustrated. The volume of the disk is

$$dV = \pi\rho^2 \; R \sin\theta d\theta$$

The mass of the disk will be

$$\begin{aligned} dm &= \frac{M}{\frac{4}{3}\pi R^3} \; dV \\ &= \frac{3M}{4R^2} \; \rho^2 \sin\theta d\theta \end{aligned}$$

The moment of inertia of the disk will be

$$dI = \frac{\rho^2 \, dm}{2} = \frac{3}{8} MR^2 \sin^5 \theta d\theta$$

where we have used $\rho = R \sin \theta$. Then the total moment of inertial will be

$$I = \frac{3}{8} MR^2 \int_0^\pi d\theta \; \sin^5 \theta = \frac{2}{5} MR^2$$