Project-Chaotic rotation of irregular moons

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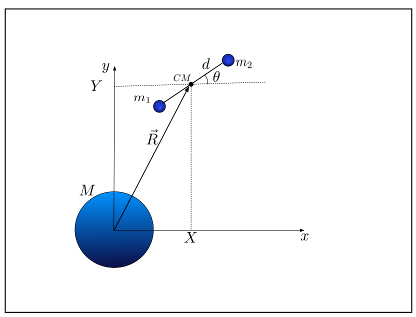
In a uniform gravitational field, torque due to gravity about the centre of mass is zero. Therefore, the spin angular momentum about the centre of mass is conserved. However, for celestial objects in orbit about other massive objects (such as a moon around a planet), the variation in gravitational field across the object results in a non-zero torque about the centre of mass. This effect, however small, can produce a change in the spin of the object over long enough time scales. In partcular, if the object is irregular in shape, the effect of this torque can be *chaotic*. This project explores this phenomenon by assuming there is an irregularly shaped moon in orbit around a planet, and aims to study the variation in the rotational motion of the moon as a function of the eccentricity of its orbit around the planet. One of Saturn's moons, Hyperion, shows such a chaotic behaviour.
For simplicity, we will model the moon as two masses connected by a (massless) rod of length d, in orbit around a planet of mass M. This artificial shape models the irregularity in the shape of the moon, responsible for the chaotic behaviour. Following is the system, along with the coordinate system used. The centre of mass has coordinates X and Y and the rod is at (instantaneous) angle θ relative to the x axis



1. First, we simulate the motion of the centre of mass of the moon about the planet. Write differential equations for the X and Y coordinates of the centre of mass. Next, reduce them to dimensionless form by choosing suitable length and time scales. The perihelion distance and eccentricity of orbit of moons and planets is usually tabulated and can be easily found. We will therefore assume we know these, and express everything in terms of these known quantities. The natural length scale is therefore the perihelion distance R_P of the moon (actually, its centre of mass). As time scale, we choose the period of the orbit. Show that the period of orbit is related to R_P and eccentricity as

$$t_0^2 = \frac{4\pi^2}{GM}\frac{R_P^3}{(1-\epsilon)^3}$$

Using these scales, show that the equations for coordinates of the centre of mass reduce to

$$\begin{aligned}\frac{d^2x}{d\tau^2} &= -\frac{4\pi^2}{(1-\epsilon)^3}\frac{x}{r^3}\\ \frac{d^2y}{d\tau^2} &= -\frac{4\pi^2}{(1-\epsilon)^3}\frac{y}{r^3}\end{aligned}$$

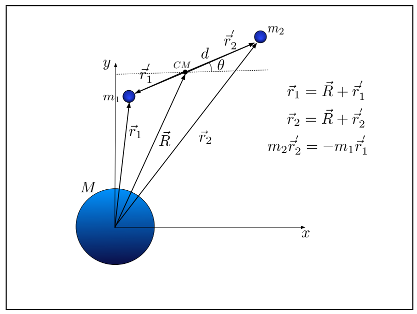
where x, y, r are dimensionless coordinates and radial distance and τ is the dimensionless time.
Now, we need initial conditions to simulate the orbit of the moon. We take the initial position to be

that at perihelion and the initial velocity as velocity at that point. First, show that the (dimensional) perihelion speed V_P is given by

$$V_P = \sqrt{\frac{GM}{R_P}} \times \sqrt{1+\epsilon}$$

What is the dimensionless perihelion speed? Choosing initial coordinates (and components of initial velocity) of centre of mass wisely, use a Verlet algorithm to determine the position and velocity of the moon as functions of time, and plot the orbit for different eccentricities. Check for energy conservation.

2. Next, we analyze the rotational motion about the centre of mass. We assume that the mass-rod system is in the plane of the orbit. Refer to the following illustration



Show that the torque about the centre of mass is

$$\begin{aligned}
\vec{\tau} &= -GMm_1(\vec{r}_1' \times \vec{R})\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) \\
&\simeq -\frac{GMm_1}{R^6}(\vec{r}_1' \times \vec{R})\left(r_2^3 - r_1^3\right) \\
&\simeq -\frac{3GMm_1}{R^4}(\vec{r}_1' \times \vec{R})(r_2 - r_1) \\
&= -\frac{3GM\mu}{R^4}(\vec{d} \times \vec{R})(r_2 - r_1)
\end{aligned}$$

where $\vec{d} = \vec{r_1} - \vec{r_2} = \vec{r}_1' - \vec{r}_2'$ and μ is the reduced mass. Next, write r_1 and r_2 as

$$\begin{aligned}
r_1 &= \sqrt{\vec{r_1} \cdot \vec{r_1}} \\
&= \sqrt{\left(\vec{r}_1' + \vec{R}\right)^2} \\
r_2 &= \sqrt{\vec{r_2} \cdot \vec{r_2}} \\
&= \sqrt{\left(\vec{r}_2' + \vec{R}\right)^2}
\end{aligned}$$

Using binomial approximation, show that

$$r_2 - r_1 \simeq -\frac{\vec{d} \cdot \vec{R}}{R}$$

so that the toque expression is

$$\vec{\tau} \simeq \frac{3GM\mu}{R^5}(\vec{d} \times \vec{R})(\vec{d} \cdot \vec{R})$$

Let $\hat{k}$ be a unit vector along the z direction (out of the plane of the illustration). Using

$$I\frac{d(\vec{\omega} \cdot \hat{k})}{dt} = \vec{\tau} \cdot \hat{k}$$

and expressing $\vec{d}$ and $\vec{R}$ in terms of their components and unit vectors $\hat{\imath}$ and $\hat{\jmath}$, show that the angular velocity (component along $\hat{k}$) satisfies the equation

$$\begin{aligned}
\frac{d\omega}{dt} &= -\frac{3GM}{R^5}\left(X\sin\theta - Y\cos\theta\right)\left(X\cos\theta + Y\sin\theta\right) \\
\implies \frac{d^2\theta}{dt^2} &= -\frac{3GM}{R^5}\left(X\sin\theta - Y\cos\theta\right)\left(X\cos\theta + Y\sin\theta\right)
\end{aligned}$$

Using our natural length and time scales, show that this reduces to

$$\frac{d^2\theta}{d\tau^2} = -\frac{12\pi^2}{(1-\epsilon)^3}\,\frac{1}{r^5}\left(x\sin\theta - y\cos\theta\right)\left(x\cos\theta + y\sin\theta\right)$$

This can be written as

$$\begin{aligned}
\frac{d\theta}{d\tau} &= \Omega \\
\frac{d\Omega}{d\tau} &= -\frac{12\pi^2}{(1-\epsilon)^3}\,\frac{1}{r^5}\left(x\sin\theta - y\cos\theta\right)\left(x\cos\theta + y\sin\theta\right)
\end{aligned}$$

where Ω is the dimensionless angular velocity.

3. Finally, we have the following differential equations for the position and velocity of the centre of mass, and the angular position and angular velocity about the centre of mass

$$\begin{aligned}
\frac{dx}{d\tau} &= v_x \\
\frac{dv_x}{d\tau} &= -\frac{4\pi^2}{(1-\epsilon)^3}\frac{x}{r^3} \\
\frac{dy}{d\tau} &= v_y \\
\frac{dv_y}{d\tau} &= -\frac{4\pi^2}{(1-\epsilon)^3}\frac{y}{r^3} \\
\frac{d\theta}{d\tau} &= \Omega \\
\frac{d\Omega}{d\tau} &= -\frac{12\pi^2}{(1-\epsilon)^3}\,\frac{1}{r^5}\left(x\sin\theta - y\cos\theta\right)\left(x\cos\theta + y\sin\theta\right)
\end{aligned}$$

Thinking of x, y and θ as position coordinates and v_x, v_y, ω as velocity components, these equations are such that the acceleration components depend only on the position coordinates. Therefore, given $x(0), y(0), \theta(0), v_x(0), v_y(0), \Omega(0)$, we can predict these quantities at all instants of time. For different values of eccentricity and the same initial conditions for centre of mass as before, choose some initial value of θ and ω and calculate them as functions of time, using the Verlet algorithm. *You will need to restrict θ to interval $\theta \in [0, 2\pi[$. Which means that in the algorithm, you need to check if θ exceeds 2π. If it does, it has to be reset to zero, just as it exceeds 2π.* Plot the variation in θ and Ω vs time for different values of eccentricity, starting with $\epsilon = 0$. Do you observe a regularity in the plots for zero eccentricity? How does the regular behaviour change as you increase the eccentricity?

4. **Chaos:** In a qualitative way, chaos can be thought of as extreme sensitivity of a system to initial conditions. A very tiny change in initial conditions can get magnified into an exponentially large difference in the behaviour of the system with time. Observe what happens if you change the initial angle and/or angular velocity by a small amount. For two such close initial conditions, observe what happens to $\theta(\tau)$ and $\Omega(\tau)$ as time progresses. Do the two sets of plots look close? If they differ substantially, after how long (approximately) do the differences become substantial?